ACTUARIAL AND FINANCIAL VALUATIONS OF GUARANTEED ANNUITY OPTIONS

by

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A PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department
of
Statistics and Actuarial Science

SIMON FRASER UNIVERSITY

Summer 2007

APPROVAL

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Title of Project: Actuarial and Financial Valuations of Guaranteed Annuity Options

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ABSTRACT

Guaranteed Annuity Options (GAOs) are options available to holders of certain pension policies. Under these contracts, policyholders contribute premiums into a fund managed by the insurer. At retirement, the policyholders buy life annuities at a guaranteed rate provided by the original insurer, or annuitize with another insurer. If the guaranteed annuity rates are better than the prevailing rates in the market, the insurer has to make up the difference. GAOs can be viewed as interest rate options, since retiring policyholders can choose to use the higher of the guaranteed annuity rate and the prevailing market rate. We study GAOs using two models for the interest rate; the Vasicek and the Cox-Ingersoll-Ross models. An actuarial approach is used to value the GAOs and compared with the value of a replicating portfolio.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor, Dr. Gary Parker, for his tireless help, invaluable guidance and constructive criticism throughout my graduate studies.

I also wish to express my great appreciation to Dr. Cary Chi-Liang Tsai and Dr. Derek Bingham, members of my examination committee, for their ever willing assistance and valuable comments.

It is with sincere gratitude that I acknowledge the support received from the Department of Statistics and Actuarial Science, Simon Fraser University, Canada. Financial support for this work was provided by the Canada Graduate Scholarships of Natural Sciences and Engineering Research Council of Canada (NSERC).

Finally, I would like to thank my husband Xun Cheng for his patience and support during the course of my studies.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1 INTRODUCTION

1.1 Guaranteed annuity options (GAOs)

Guaranteed Annuity Options (GAOs) are options available to holders of certain pension policies. Under these contracts, policyholders contribute either single or regular premiums into a fund managed by the insurer. At retirement, the policyholders have the option to convert the maturity policy proceeds into life annuities at a guaranteed rate provided by the original insurer, or annuitize with another insurer. If the guaranteed annuity rates are more beneficial to the policyholders than the prevailing rates in the market, the insurer has to make up the difference. GAOs can be viewed as interest rate options, since retiring policyholders can choose to use the higher of the guaranteed annuity rate and the prevailing market rate.

GAOs have been designed to make the pension contract more attractive since the policyholder could count on a minimum annuitization rate. There is evidence of GAOs being issued in 1839 (Historic Records Working Party (1972)). Today, GAO has become a common feature for many US tax sheltered insurance products. A survey conducted by the Government Actuary's Department in 1998 on life insurance companies' exposure to GAOs indicated that: the exposure to GAOs was relatively widespread within the industry and had the potential to have a significant financial effect on a number of

companies (Treasury (1998)). However, it is the GAOs of UK retirement savings contracts sold in the 1970s and 1980s that drew most of the attention.

Guaranteed Annuity Options began to be included in some UK pension policies in the 1950's and became very popular in the 1970's and 1980's when long-term interest rates were high. At that time, the GAOs were set very far out-of-the-money and insurance companies apparently assumed that interest rates would remain above the implicit guaranteed rates and consequently that the guarantee would be a cost-free benefit to make policyholders feel more secure.

In the 1970s and 1980s, the most popular guaranteed rate for a male aged sixty five was £111 annuity per annum per £1000 of maturity proceeds or an annuity cash value ratio of 1:9. If the prevailing annuity rate provides an annual payment higher than £111 per £1000, a rational policyholder would choose the prevailing market rate. During these two decades, the average UK long-term interest rate was around 11% p.a. The break-even interest rate implicit in the GAOs based on the mortality basis used in the original calculations was in the region of 5%-6% p.a. (see Boyle & Hardy (2003)). Obviously, the GAOs were far out-of-the-money. However, in the 1990's, as long-term interest rates fell, these GAOs began to move into the money. The inclusion of GAOs was discontinued in the UK by the end of 1980's. Unfortunately, the long-term nature of these pension policies still made GAO a significant risk management challenge for the life assurance industry and threatened the solvency of some UK insurance companies. The emerging liabilities under GAOs (near £2.6 billion) forced Equitable life (UK), the world's oldest mutual insurance company, to stop issuing new business in 2000.

When these GAOs were written in the 1970s and 1980s, although most actuaries believed that long-term interest rates would hardly fall below the break-even points, it is quite possible that actuaries were aware of a distant tail risk. At that time, the insurers were sitting on huge reserves of "free assets" known as the estate. The ratio of assets to statutory liabilities was usually greater than 150 percent, and this gave the insurer a financial position deemed safe enough to handle the tail risk. Unfortunately, the political climate changed in the late 1980s. The excess assets of insurers were viewed as cross-generational subsidy which was inequitable and had to be distributed to policyholders. The company that most dramatically and openly championed the minimal estate approach was Equitable Life (UK).

In 1995, when Equitable Life realized that the guaranteed annuity options were moving into the money, they sought to recover the cost of meeting the annuity guarantees, where claimed, by reducing what would otherwise have been the policyholder's final bonus (Report of the Equitable Life Enquiry (2004)). They believed that the life insurance company had a wide discretion as to how to set future discretionary bonuses. The policyholder's right to convert the proceeds of his policy on maturity into an annuity at a pre-determined rate was a contractual right that could not be denied, but the actual amount of the proceeds at maturity depended on how the company used its discretion under the discretionary profit. However, Equitable Life's use of its discretion in this way was litigated through the UK court system and ultimately reached the highest court in the UK, the House of Lords. At the level of junior courts, Equitable Life found some support for their view, but when the court reached the House of Lords the judges

unanimously held that Equitable Life's discretion in setting terminal bonus was not unfettered. It needed to be exercised fairly and for a proper purpose. Using that discretion to defeat the purpose of another contractual term, the guaranteed annuity option, was not a proper purpose. The House of Lords ruled that Equitable Life had to meet its obligations to its policyholders with annuity guarantees. The cost of this decision sufficiently undermined the Equitable Life's financial position that the mutual had to put itself for sale in July 2000. After failing to find a buyer, Equitable Life closed its doors to new business in December 2000.

Besides the decline in long-term interest rate, two other factors also contributed to the dramatic increase in the liabilities associated with these guarantees. First, strong stock market performance during the last two decades of the twentieth century meant that the amounts to which the guarantee applied increased significantly. Between 1980 and 2000, the annualized rate of growth on the major UK stock index was around 18 percent per year. Second, the mortality assumption implicit in the guarantee did not anticipate the significant improvement in mortality which occurred during that period. Comparing the mortality tables used during the 1970s to the current ones, it is evident that mortality improvements in the UK have increased the life expectancy of a 65-year-old man by about five years. Clearly, additional life expectancy increases the value of the life annuity for any given interest rate.

1.2 Literature review

Guaranteed annuity options have drawn considerable publicity in recent years. Bolton et al. (1997) described the origin and nature of these guarantees. Boyle (2003) analyzed their pricing and risk management. O'Brien (2001) discussed issues arising from GAOs in pension policies issued by U.K. life assurance companies and highlighted the impact of improving mortality. Many researchers have applied either actuarial methods or no-arbitrage pricing theory to calculate the value of GAOs embedded in deferred annuity pension policies. In Pelsser (2002), a market value for GAO was derived using martingale modelling techniques and a static replicating portfolio of vanilla interest rate swaptions that replicates the GAO was constructed. The replicating portfolio would have been extremely effective and fairly cheap as a hedge against the interest rate risk involved in the GAO based on the UK interest rate data from 1980 until 2000. Chu & Kwok (2007) proposed three analytical approximation methods for the numerical valuation of GAOs: stochastic duration approach, Edgeworth expansion and analytic approximation in affine diffusions. In Chu & Kwok's work, a two-factor affine interest rate term structure model was used. Ballotta and Haberman (2003) applied the one-factor Heath-Jarrow-Morton model to price GAOs in unit-linked deferred annuity contracts with a single premium. In Boyle & Hardy (2003) a simple one-factor interest rate model was used and the market price of the GAOs were obtained by option pricing approach. Boyle and Hardy also examined a number of conceptual and practical issues involved in dynamic hedging of the interest rate risk. Wilkie et al. (2003) worked on unit-linked contracts and investigated two approaches to reserving and pricing. Their first approach is traditional actuarial approach: quantile, conditional tail expectation and reserves. The

second approach is to use option pricing methodology to dynamically hedge a guaranteed annuity option. The 1984 and 1995 Wilkie models were used to depict the yield curve.

1.3 Outline of the project

The work presented in this project is based on Boyle & Hardy (2003) and Wilkie et al. (2003). However, instead of the Wilkie models, the interest rate dynamics are modelled by Vasicek and Cox-Ingersoll-Ross (CIR) models. The rest of the project is organized as follows. In the next section, the model setup of the GAO is presented. Vasicek and CIR models are introduced and estimated in Section 3. Maximum likelihood estimator is applied for the Vasicek model while approximated and exact Gaussian estimations are used for the CIR model. The validity of these estimation methods is also examined in Section 3. In Section 4, the actuarial approach to value the GAOs is investigated. Monte Carlo simulation is used to derive the distribution of the guaranteed annuity options and the percentiles as well as VaRs (value at risk) are thus determined. The option pricing and hedging approach is studied in Section 5. A replicating portfolio consisting of equities and zero-coupon bonds is constructed to replicate the GAO and simulation results of delta hedging are presented. The sensitivity of the value of the guaranteed annuity option with respect to different parameters in the pricing model is also investigated. Conclusions and areas of future work are given in section 6.

2 MODEL SETUP OF THE GAO

For simplicity, only single premium equity-linked policies are considered in this project. Like in Boyle & Hardy's paper (Boyle & Hardy (2003)), standard actuarial notations are used in setting up the GAO model. Assume a male purchases a single-premium equity-linked contact and pays the premium at time 0. The contract will mature at time *T*, say, at which date the policyholder will reach age 65. The premium is invested in an equity account with market value *S(t)* at time *t*, where *S(t)* is a random process. *S(t)* is derived from the value of shares, with dividends reinvested. Thus, at time *T*, the policyholder will have maturity proceeds of *S(T)*.

At maturity, the market cost of a life annuity of \$1 per annum for a male aged 65 is denoted by $a_{65}(T)$. Note that $a_{65}(T)$ is also a random process and is determined by the mortality assumption, the expense assumption and the interest rate dynamics at time *T*. Therefore, if the policyholder purchases an annuity at market rates, the annuity that could be received is $S(T)/a_{65}(T)$ per annum, and the market value of this annuity is just $S(T)/a_{65}(T) \times a_{65}(T) = S(T)$.

The policy offers a guaranteed annuity rate of $g = 9$, that is, \$1 of lump sum maturity value purchases \$1/*g* of annuity per annum. With *S(T)* maturity proceeds, the policyholder is able to purchase an annuity of $S(T)/g$, which has a market value of

$\left(S(T)/g\right)\times a_{65}(T)$. A rational policyholder will choose whichever is higher: i.e. $\max\left(S(T)/g\times a_{65}(T),\quad S(T)\right)$. The insurer will cover the excess of the annuity cost over the maturity proceeds:

$$\max\left(S(T)/g\times a_{65}(T),\quad S(T)\right)-S(T)$$

or

$$S(T)\times\max\left(a_{65}(T)/g-1,\quad 0\right). \tag{2.1}$$

$a_{65}(T)$ is the market value of a life annuity of \$1 payable annually in arrear starting at time T for a male aged 65. It depends on the prevailing long-term interest rates, the mortality assumption, and the expense assumption. In the project, the expenses are ignored and the mortality risk is assumed to be fully diversified. The value of $a_{65}(T)$ is given by

$$a_{65}(T)=\sum_{n=1}^{\omega-65}{}_{n}p_{65}\times D_{T+n}(T) \tag{2.2}$$

where ${}_{n}p_{r}$ is the probability that a life aged r survives n years and $D_{T+n}(T)$ denotes the market value at time T of the unit par default-free zero-coupon bond with maturity date $T+n$. The limiting age of the policyholder is denoted by ω.

The policyholder's death may occur between time 0 and time T. It is assumed that on death the only benefit is a return of the value of the fund, $S(t)$. As a result, a proportion

of only ${}_T p_{65-T}$ policies, where ${}_T p_{65-T}$ is calculated using an appropriate mortality table, survives to maturity. Since only those who survive to age 65 can receive the annuity, the value of the GAO per initial life is reduced to ${}_T p_{65-T} \times S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right)$.

Let

$$V(T) \quad = \quad {}_T p_{65-T} \times S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right) \tag{2.3}$$

be the value of the GAO per initial life at maturity time *T*. The GAO valuation problem thus becomes to find *V(t)*, the discounted value of *V(T)* at time *t*, $0 \le t \le T$.

It must be noted that a few assumptions have been made in the project:

1. The expenses are ignored when calculating $a_{65}(T)$.

2. The premiums are invested and the annuities are purchased in US market.

3. The equity and bonds are uncorrelated.

4. The mortality risk has been fully diversified and is independent of the financial risk.

3 INTEREST MODELS AND THEIR ESTIMATIONS

The value of the GAO at maturity is given by equation (2.1). From this equation, it can be observed, that two stochastic processes *S(T)* and $a_{65}(T)$ determine the market value of the GAO.

In addition, under the assumptions of no expense and fully diversified mortality risk, $a_{65}(T)$ is solely determined by the prevailing mortality rates and interest rates.

Thus, the first step in a GAO valuation is to find appropriate interest rate models and estimate their parameters using historical interest rate data. The interest rate models employed in the project are the Vasicek model and the Cox-Ingersoll-Ross (CIR) model. Both models are continuous-time one-factor short-rate models.

Studies found that multifactor models generally outperform one-factor ones over longer forecast horizons. However, as suggested by Hull (2002), relatively simple one-factor models usually give reasonable prices for instruments if used carefully. Moreover, compared to multi-factor models, one-factor models lead to more straightforward closed-form formula for the GAO.

3.1 Vasicek model

The Vasicek model is one of the earliest stochastic term structure models (Vasicek (1977)). It can be formulated in terms of a linear stochastic differential equation of its short-term interest rate *r(t)*:

$$dr(t) = \kappa(\mu - r(t))dt + \sigma dW_t^P \tag{3.1.1}$$

where W_t^P is a standard Brownian motion (or Wiener process) under the real-world measure P and κ, μ, and σ are unknown system parameters. Under the Vasicek model, *r(t)* reverts towards the unconditional mean μ, κ measures the speed of mean reversion (the larger κ, the faster the speed of mean reversion), and σ is the instantaneous volatility of the short-rate *r(t)*. If *r(t)* is above the mean ($r(t) > \mu$), then the coefficient κ (>0) makes the drift become negative so that the rate will be pulled down in the direction of μ. Likewise, if the rate is below the mean, the coefficient κ (>0) makes the drift become positive and the rate is pulled up in the direction of μ. This stochastic process is also known as the Ornstein-Uhlenbeck process.

Under the risk-neutral measure Q, the short-rate *r(t)* evolves according to the SDE

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW_t^Q \tag{3.1.2}$$

where $\theta = \mu - \lambda\sigma/\kappa$ is the risk-neutral mean and W_t^Q is a Brownian motion under the risk-neutral measure Q. The solution to the SDE (3.1.2) can be written as

$$r(t) = r(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s^Q \tag{3.1.3}$$

or, for s<t, $$r(t) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \sigma \int_s^t e^{-\kappa(t-\tau)} dW_\tau^Q \tag{3.1.4}$$

Thus, *r(t)* is Gaussian at each *t*, with expectation:

$$E^Q[r(t)] = r(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}) \tag{3.1.5}$$

and variance

$$V^Q[r(t)] = \sigma^2 E\left[\left(\int_0^t e^{-\kappa(t-s)} dW_s^Q\right)^2\right] = \sigma^2 \int_0^t e^{-2\kappa(t-s)} ds = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}). \tag{3.1.6}$$

It can be observed from equation (3.1.5) that the conditional expectation of the short-rate *r(t)* given *r(s)* is a weighted average of *r(s)* and its long-term mean.

In the Vasicek model the risk is captured by assuming that the market price of interest rate risk, $(\mu - r)/\sigma = \lambda$, is constant across the term structure. This assumption is essentially the same as the no-arbitage/equivalent martingale assumption and permits pricing in a risk-neural framework.

The price of a \$1 face value zero-coupon bond at time *t* with maturity date *T* is (see Cairns (2004))

$$P(t,T) = E^{Q}\left[e^{-\int_t^T r(s)ds} \;\middle|F_t\right] = A(t,T)e^{-B(t,T)r(t)}$$

$$\text{where} \quad B(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{3.1.7}$$

$$\text{and} \quad A(t,T) = \exp\left[\left(\theta - \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2\right)\left(B(t,T) - T + t\right) - \frac{\sigma^2}{4\kappa}B^2(t,T)\right].$$

The Vasicek model has several weaknesses. As a simple one-factor model, the Vasicek model cannot capture the more complex term structure shifts that occur. For s<t, *r(t)* given *r(s)* is normally distributed with mean $r(s)e^{-\kappa(t-s)} + \mu(1-e^{-\kappa(t-s)})$ and variance $\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa(t-s)})$ under the real-world measure *P*. Therefore, if *r(s)* is small, it is possible for the short-rate *r(t)* to become negative. Moreover, all short-rates have the same volatility σ. These are undesirable in real world applications.

3.2 CIR model

The CIR model for interest rates was proposed in 1985 (see Cox et al. (1985)). Since then it has been, and still is, the object of many studies and extensions. As a popular one-factor model, the CIR model has many desirable features such as supporting empirical evidence, positivity of the interest rates under certain conditions (see Cox et al. (1985)), uncomplicated fitting to data, mean reversion, interest rate dependent volatility and availability of closed-form pricing formula (Maghsoodi (2000)).

The CIR model is a particular case of affine models. Under the CIR model, the short-rate *r(t)* evolves according the SDE

$$dr(t) = \kappa(\mu - r(t))dt + \sigma\sqrt{r(t)}dW_t^P . \tag{3.2.1}$$

In SDE (3.2.1), the drift factor, $\kappa(\mu - r(t))$, is exactly the same as in the Vasicek model. It ensures mean reversion of the interest rate towards the long run value μ, with speed of adjustment governed by the strictly positive parameter κ. The diffusion factor, $\sigma\sqrt{r(t)}$, corrects the main drawback of Vasicek's model, ensuring that the interest rate cannot become negative. Thus, at low values of the interest rate, the standard deviation becomes close to zero, cancelling the effect of the random shock on the interest rate. Consequently, when the interest rate gets close to zero, its evolution becomes dominated by the drift factor, which pushes the rate upward towards equilibrium.

It must be noted that the short-rate *r(t)* in the CIR model has a non-central chi-square transition density, while the Vasicek model has a Gaussian one. The expectation and variance of *r(t)* given *r(0)* are as follows:

$$\begin{aligned} E(r(t)) &= r(0)e^{-\kappa t} + \mu(1 - e^{-\kappa t}), \\ V(r(t)) &= r(0)\frac{\sigma^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t})^2 . \end{aligned} \tag{3.2.2}$$

In CIR, the market price of risk is specified as $\lambda(r(t)) = \lambda\sqrt{r(t)}/\sigma$. The scaling by σ is done only to simplify the subsequent derivations. Let $\tau = T - t$, once again, the bond-pricing formula takes the exponential-affine form

$$P(t,T) = \exp\big(A(\tau) + B(\tau)r(t)\big) \tag{3.2.3}$$

with *B(τ)* and *A(τ)* being defined as follows (see Cairns (2004)):

$$B(\tau) = \frac{-2(1-e^{-\gamma\tau})}{2\gamma + (\kappa+\lambda-\gamma)(1-e^{-\gamma\tau})},$$

$$A(\tau) = \frac{2\kappa\mu}{\sigma^2}\log\left[\frac{2\gamma e^{(\kappa+\lambda-\gamma)\tau/2}}{2\gamma + (\kappa+\lambda-\gamma)(1-e^{-\gamma\tau})}\right], \tag{3.2.4}$$

$$\gamma = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}.$$

Both the Vasicek and CIR models are mean-reverting processes. They differ in the diffusion term, where $\sigma\sqrt{r(t)}$ gives the CIR model a 'level effect'. Indeed, empirically it is observed that volatility increases with the level of interest rates.

Different methods can be used for estimating the Vasicek and CIR models. For example, the covariance equivalence principle can be used to establish relationships between the parameters of continuous processes and those of their discrete representations. For an application of this method, see Parker (1995).

The estimation methods used for this project are described in the next two sections.

3.3 Maximum likelihood estimation for Vasicek model

The Vasicek model can be specified by either real-world parameters κ, μ, σ, λ, or risk-neutral parameters κ, θ, σ. In this project, κ, μ, and σ are estimated using maximum likelihood methods with time-series data and λ is estimated by least square method using cross-sectional data. As the short-rate, $r(t)$, of the Vasicek model evolves according to SDE (3.1.1), the corresponding exact discrete model then has the form

$$r(t) = r(t-\Delta)e^{-\kappa\Delta} + \mu(1-e^{-\kappa\Delta}) + \eta(t) \tag{3.3.1}$$

where $\eta(t)$ has the conditional distribution $\eta(t) \mid F_{t-\Delta} \sim N\left(0, \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa\Delta})\right)$. That is, the short-rate *r(t)* given *r(t-Δ)* is normally distributed and the parameters κ, μ, and σ can be estimated using maximum likelihood method.

Let $\{r_i\}_{i=0}^{n}$ be equally spaced samples of the short-rate r*(t)*, and $r_0 = r(t_0)$, $r_{i+1} = r(t_{i+1}) = r(t_i + \Delta)$. If yearly samples are collected, *Δ* is 1. For monthly, weekly, and daily samples, *Δ* equals to 1/12, 1/52, and 1/250 respectively. The likelihood function of the Vasicek model and the corresponding log-likelihood can thus be written as

$$L(\kappa, \mu, \sigma) = \prod_{i=1}^{n} f(r_i \mid r_{i-1}),$$

$$l(\kappa, \mu, \sigma) = \sum_{i=1}^{n} \ln f(r_i \mid r_{i-1}).$$

Using Nowman's notation (Nowman (1997))

$$m_{ii}^2 = V(r_i \mid r_{i-1}) = \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa\Delta}) \tag{3.3.2}$$

and the transition log-probability is

$$\ln f(r_i \mid r_{i-1}) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(m_{ii}^2) - \frac{[r_i - r_{i-1}e^{-\kappa\Delta} - \mu(1-e^{-\kappa\Delta})]^2}{2m_{ii}^2}.$$

Therefore, the log-likelihood function for the Vasicek model is

$$l(\kappa,\mu,\sigma) \propto \sum_{i=1}^{n}\left\{-\frac{1}{2}\ln(m_{ii}^2) - \frac{\left[r_i - r_{i-1}e^{-\kappa\Delta} - \mu\left(1-e^{-\kappa\Delta}\right)\right]^2}{2m_{ii}^2}\right\} \qquad (3.3.3)$$

where m_{ii}^2 is given in equation (3.3.2). The parameters κ, μ, and σ are then found by maximizing the log-likelihood (3.3.3). The market price of risk λ is identified by the least square method for cross-sectional data. In our experiments, US long-term interest rates (10-year and 20-year rates) are used to calculate the historical prices of the corresponding \$1 face value zero-coupon bonds, *P(0,10)* and *P(0,20)*. Our estimate for λ is the value that minimizes the square error between these historical bond prices and the bond prices given by equation (3.1.7) with the MLE for κ, μ, and σ.

Note that the maximum likelihood here is not a true maximum likelihood but a "discretized maximum likelihood". As $\Delta \to 0$, the sample paths of the discretization (3.3.1) converge to the continuous path (3.1.1). However, as the data *r(t)* can only be recorded with certain minimum intervals, the "discretized maximum likelihood estimator" for κ, θ, and σ will not be consistent (Miscia (2004)).

To test the validity and evaluate the finite sample performance of the estimation method with finite samples, a small Monte Carlo study similar to Yu & Philips's (Yu & Philips (2001)) is conducted. The SDE (3.1.1) of the interest rate *r(t)* is rewritten as

$$dr(t) = (\alpha + \beta r(t))dt + \sigma dW_t^P .$$

Obviously, $\alpha=\kappa\mu$ and $\beta = -\kappa$. Given $\alpha = 6.0$, $\beta = -1.0$, $\sigma = 0.25$, and initial value $r_0 = 7.0$, 2000 daily rates are generated using equation (3.3.1). It must be noted that it is

the discrete model (3.3.1) (not the continuous model) that is tested. The Vasicek model is fitted to the simulated sequence by the maximum likelihood method described above. The experiments are repeated for 1000 times and the means and variances of the resultant estimates are given in Table 3.1.

Table 1: Monte Carlo study of maximum likelihood estimation method for daily data

	α	β	σ
Mean	7.325675	-1.215805	0.2497247
variance	6.105208	0.1597621	1.555029e-05

The means of the estimates for α and β are 7.325675 and -1.215805 respectively, They are different from the parameters' true values. However, the estimated long-term mean $\mu = -\frac{\alpha}{\beta}$ has a value of 6.02, which is very close to the true value of 6.0. The mean of the estimates for σ is 0.2497247, slightly below the true value 0.25. The variance of the estimates for σ is very small: 1.555029e-05. This indicates that the maximum likelihood method is able to produce very good estimates of σ and the long-term mean, μ. When it comes to the estimates of α and β, however, the results are relatively poor.

Table 2 gives the corresponding results for weekly data. The sample size of weekly data is 1000. For weekly data, the maximum likelihood method still gives very good estimate of σ, pretty good estimate of the long-term mean, and reasonable estimates of α and β. Comparing Tables 1 and 2, it can be observed that all the results in Table 1 are better than those of Table 2. The results verify that as $\Delta \to 0$, the sample paths of the

discretization converge to the continuous path, the discretized maximum likelihood estimators tend to be consistent.

Table 2: Monte Carlo study of maximum likelihood estimation method for weekly data

	α	β	σ
Mean	7.82823	-1.293266	0.2499380
variance	10.82493	0.2735813	3.001517e-05

The short-rate *r(t)*, when observed at the daily, weekly and even monthly frequencies, tends to have large autoregressive coefficients. The autocorrelation properties of the sequence $\{r_i\}$ are determined by the parameter β. It is well known that the ML estimate of the autocorrelation parameter for a sequence that almost has a "unit root" is downward biased (Andrews (1993)). Therefore, the ML estimate of β will have a downward bias which will result in an upward bias in the estimate of α. This is consistent with the numbers in Tables 1 and 2.

3.4 Approximated Gaussian estimation for CIR model

The SDE as given in equation (3.2.1) shows that the absolute variance of the interest rate in the CIR model increases when the interest rate itself increases. As a result, the transition density of r_i given r_{i-1} is not Gaussian. To find a Gaussian approximation, equation (3.2.1) is approximated by the following SDE in Nowman's paper (Nowman (1997)).

$$dr(t) = \kappa(\mu - r(t))dt + \sigma\sqrt{r(t'-1)}dW_t^P$$
$$or \quad dr(t) = (\alpha + \beta r(t))dt + \sigma\sqrt{r(t'-1)}dW_t^P$$

where $t'-1$ corresponds to the largest sample point less than t. That is, with sample data $\{r_0 = r(t_0),\ r_1 = r(t_1),\ \ldots\ ,\ r_i = r(t_i),\ r_{i+1} = r(t_{i+1}),\ \ldots\ ,\ r_n = r(t_n)\}$ and $t_i < t < t_{i+1}$, we have $t'-1 = t_i$ and $r(t'-1) = r_i$. By using the above approximation, it is assumed that the volatility of the interest rate changes at the beginning of the observation period and then remains constant. The short-rate $r(t)$ then satisfies the following stochastic integral equation

$$r(t) - r(t_i) = \int_{t_i}^{t} [\alpha + \beta r(s)] ds + \sigma \sqrt{r(t_i)} \int_{t_i}^{t} dW_s^P \tag{3.4.1}$$

for all t in $(t_i, t_{i+1}]$. From Bergstrom's Theorem 2 (Bergstrom, 1982), the discrete model corresponding to equation (3.4.1) is given by

$$r(t_i) = r(t_{i-1}) e^{-\kappa\Delta} + \mu(1 - e^{-\kappa\Delta}) + \eta_i \qquad (i = 1, \ldots, n) \tag{3.4.2}$$

where η_i $(i = 1, \ldots, n)$ satisfies the conditions $E(\eta_i \eta_j) = 0$ $(i \neq j)$ and

$$E\left(\eta_i^{\,2}\right) = \int_{t_{i-1}}^{t_i} e^{-2\kappa(t_i - s)} \sigma^2 \left(\sqrt{r(t_{i-1})}\right)^2 ds = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta}) r(t_{i-1}) = m_{ii}^2 . \tag{3.4.3}$$

Here, η_i is approximated by a normal distribution with mean 0 and variance m_{ii}^2. The log-likelihood function of the CIR model can be approximated by the following Gaussian log-likelihood function

$$l(\kappa, \mu, \sigma) \propto \sum_{i=1}^{n} \left\{ -\frac{1}{2} \ln(m_{ii}^2) - \frac{[r_i - r_{i-1} e^{-\kappa\Delta} - \mu(1 - e^{-\kappa\Delta})]^2}{2 m_{ii}^2} \right\} \tag{3.4.4}$$

where m_{ii}^2 is given in equation (3.4.3). The above estimation method was introduced by Nowman and therefore will be called the Nowman method in this project.

The Nowman method can be understood as using the Euler method to approximate the diffusion term over the interval. Compared with the discretization method where the Euler method is applied to both the drift and diffusion terms in the diffusion process, the Nowman method can be expected to reduce some of the temporal aggregation bias. Strictly speaking, the method is a form of quasi-maximum method since (3.4.2) is not a true discrete model corresponding to equation (3.2.1) but is merely a conditional Gaussian approximation (Yu & Philips (2001)).

Once we have the log-likehood function, estimates for parameters κ, μ, and σ are found by maximizing (3.4.4). The market price of risk, λ, is estimated by the least square method using cross-sectional data as explained in the previous section.

A Monte Carlo study similar to Yu & Philips's is conducted to test the validity of the approximated Gaussian estimation method. The SDE (3.2.1) of the interest rate *r(t)* is rewritten as

$$dr(t) = (\alpha + \beta r(t))dt + \sigma\sqrt{r(t)}dW_t^P .$$

Given $\alpha = 6.0$, $\beta = -1.0$, $\sigma = 0.25$, and initial value $r_0 = 7.0$, 2000 daily data r_i's are simulated using equation (3.4.2). Note that the discrete model (3.4.2) instead of the continuous model (3.2.1) is used in the simulation. The discrete version of the CIR model is fitted to the simulated sequence by the approximated Gaussian estimation

method described above. The experiment is repeated for 1000 times and the results (which are very similar to those of Yu & Philips's) are given in Table 3.

Table 3: Monte Carlo study of estimated Gaussian method for weekly data

	α	β	σ
Mean	8.87902	-1.466654	0.2555942
variance	16.87403	0.4347623	0.02752131

The experiment results show that the Nowman (or approximated Gaussian) method can give a quite accurate estimate of the parameter σ. Like the ML estimates, the estimate for α is upward biased and the estimate for β has a downward bias.

3.5 Exact Gaussian estimation for CIR model

Yu & Philips (2001) developed an exact Gaussian estimation for the CIR model by applying Dambis, Dubins-Schwarz theorem (hereafter DDB theorem) (Revuz & Yor (1999)). The approach is based on the idea that any continuous time martingale can be written as a Brownian motion after a suitable time change.

Lemma (DDB Theorem) *Let M be a (F_t, P) - continuous local martingale vanishing at 0 with quadratic variation process $[M]_t$ such that $[M]_\infty = \infty$. Set*

$$T_t = \inf\{s \mid [M]_s > t\}$$

then, $B_t = M_{T_t}$ *is a* (F_{T_t}) *- Brownian motion and* $M_t = B_{[M]_t}$.

The process B_t is referred to as the DDB Brownian motion of M. According to this result, when the chronological time in the local martingale M is adjusted to time T_t, the process is transformed to a Brownian motion.

From the SDE (3.2.1), the solution for $r(t+h)$ for any $h > 0$ given $r(t)$ is

$$r(t+h) = r(t)e^{-\kappa h} + \mu(1-e^{-\kappa h}) + \int_0^h \sigma e^{-\kappa(h-s)}\sqrt{r(t+s)}dW_s^P . \qquad (3.5.1)$$

Let $M(h) = \sigma \int_0^h e^{-\kappa(h-s)}\sqrt{r(t+s)}dW_s^P$. $M(h)$ is a continuous martingale with quadratic variation

$$[M]_h = \sigma^2 \int_0^h e^{-2\kappa(h-s)} r(t+s)ds . \qquad (3.5.2)$$

The time transform in the lemma is used to construct a DDB Brownian motion to represent the process $M(h)$. To do so, a sequence of positive numbers $\{h_j\}$ that produces the required time changes is introduced. For any fixed constant $a > 0$, let

$$h_{j+1} = \inf\{s \mid [M_j]_s \geq a\} = \inf\{s \mid \sigma^2 \int_0^s e^{-2\kappa(s-\tau)} r(t_j+\tau)d\tau \geq a\} \qquad (3.5.3)$$

and construct a sequence of time points $\{t_j\}$ using the iterations $t_{j+1} = t_j + h_{j+1}$ with t_1 assumed to be 0. Evaluating equation (3.5.1) at $\{t_j\}$, one gets

$$r(t_{j+1}) = r(t_j)e^{-\kappa h_{j+1}} + \mu(1-e^{-\kappa h_{j+1}}) + M(h_{j+1}) . \qquad (3.5.4)$$

According to the lemma, $M(h_{j+1}) \sim N(0, a)$. Hence, equation (3.5.4) is an exact discrete model with Gaussian noises that can be estimated directly by maximum likelihood using

$$l(\kappa,\mu,\sigma) \propto \sum_j \left\{ -\frac{1}{2}\ln(a) - \frac{\left[r(t_{j+1}) - r(t_j)e^{-\kappa h_{j+1}} - \mu\left(1-e^{-\kappa h_{j+1}}\right)\right]^2}{2a} \right\}. \qquad (3.5.5)$$

It must be noted that the sequence $\{t_j\}$ is determined by the parameters σ and κ or β if the SDE (3.2.1) is written in the form

$$dr(t) = (\alpha + \beta r(t))dt + \sigma\sqrt{r(t)}dW_t^P .$$

Since interest rates are always observed at discrete time intervals in practice, the time-change formula (3.5.3) cannot be applied directly. Instead, the following discrete time approximation is used:

$$h_{j+1} = \Delta \min\left\{ s \mid \sum_{i=1}^{s} \sigma^2 e^{2\beta(s-i)\Delta} r(t_j + i\Delta) \geq a \right\}. \qquad (3.5.6)$$

To use the exact Gaussian method, a value for a must be selected. Asymptotically, the choice of a should not matter as long as a is finite, but the same is not true in finite samples. If a is chosen too large, then the effective sample size will be too small. If a is too small, every data point will be selected and the time-change procedure becomes useless. Yu & Philips (2001) suggested to use the ML estimate, say $\hat{a}$, of the unconditional volatility of the error term of the following model

$$r(t) = r(t-\Delta)e^{\beta\Delta} + \frac{\alpha}{\beta}(e^{\beta\Delta} - 1) + \varepsilon_t$$

with $\varepsilon_t \sim N(0, a)$. However, studies show that the residual error terms are very small and cannot be used as an estimate for a. A value is selected for a in our experiments by trial and error.

Implementation of the exact Gaussian method thus proceeds as follows:

(1) select a value for a by trial and error;

(2) perform the Nowman (approximated Gaussian) estimation to obtain $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\sigma}$;

(3) set σ to $\hat{\sigma}$ since the Nowman estimate of σ is quite good in finite samples;

(4) start with α and β set to $\hat{\alpha}$ and $\hat{\beta}$ respectively and modify them in the subsequent steps;

(5) choose $\{h_j\}$ according to the time change formula (3.5.6);

(6) find the values of α and β which maximize the likelihood function (3.5.5), modify α and β, go back to step (5) until the termination condition is reached.

Monte Carlo studies are carried out to validate the exact Gaussian estimation method. 2000 simulated daily interest rates are generated by using equation (3.4.2) and the CIR model is fitted. Parameter values of $\alpha = 6.0$, $\beta = -1.0$, $\sigma = 0.25$, and initial value $r_0 = 7.0$ are arbitrarily chosen. The experiment is repeated 1000 times and the final results are tabulated in Table 4.

Table 4: Monte Carlo study comparing Nowman's method and the exact Gaussian method for daily data

	Nowman's method			Exact Gaussian method		
	α	β	σ	α	β	σ
Mean	8.87902	-1.466654	0.2555942	8.692365	-1.446294	0.2555942
Variance	16.87403	0.4347623	0.02752131	16.89037	0.4491793	0.02752131

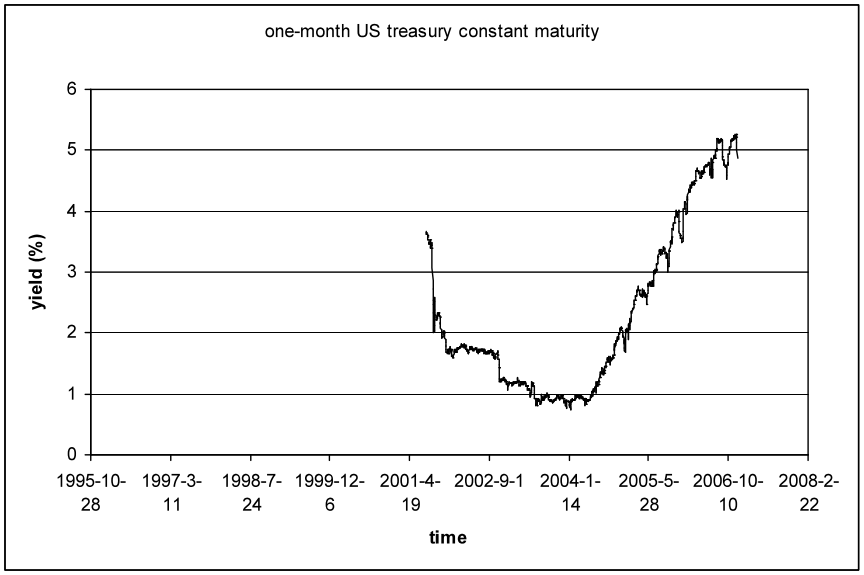
From Table 4, it can be observed that the upward and downward biases for α and β are still present in the exact Gaussian estimates. However, the biases are smaller than those of Nowman's method. Similar improvements can also be found when weekly and monthly data are used. This suggests the efficiency of the exact Gaussian estimation method. In the following section, the models and approximations described in the above sections are employed and their estimates using US treasury rates are presented.

3.6 Model estimations using historical interest rate data

Both the Vasicek and CIR models are formulated in terms of SDEs of their short-rates. As the short-rate *r(t)* is unobservable in the market, a proxy has to be used. Short-term interest rates are frequently used as a proxy. For example, Anderson and Lund (1997) and Stanton (1997) used the yield on a three-month Treasury bill as a proxy for the short-rate, while Chan et al. (1992) use the yield on one-month Treasury bill. Chapman et al. (1999) analyzed the impact of using one- or three-month yields as proxies and found that for single-factor affine models in general, the economic significance of the parameter estimate errors and the resulting bond price errors are generally negligible.

In the project, two series of short-term interest rates are considered. They are one-month and three-month US treasury constant maturity yields, obtained from Federal Reserve Statistical Release (http://www.federalreserve.gov/Release/h15/data.htm#fn10). The one-month US treasury rate series contains 1336 daily observations over the period from 31/07/2001 to 05/12/2006 which are shown in Figure 1.

Figure 1: One-month US treasury constant maturity yields

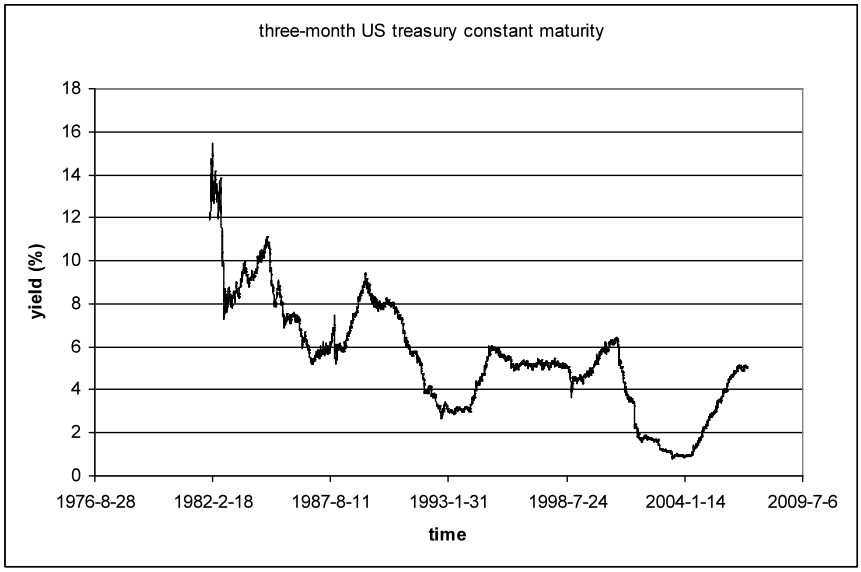


From the introductions in sections 3.1 and 3.2, we know that both the Vasicek and CIR models have long-term means and exhibit mean-reversion characteristics. In figure 1, however, no long-term mean can be observed. The interest rate first goes down and then goes up to a record high in December 2006. When fitting the Vasciek model, the ML estimation gave a bizarre result of $\beta = 0.094445505$. With β being positive, the process is not converging but diverging. This is obviously unacceptable. Thus, it can be concluded

that the one-month US treasury data set (from 31/07/2001 to 05/12/2006) should not be used for the estimation.

The three-month US treasury rate data collected covers the period from 04/01/1982 to 05/12/2006. It contains 6231 daily observations that are graphed in Figure 2.

Figure 2: Three-month US treasury constant maturity yields



It can be observed from the figure that the interest rate went down from about 15% in 1982 to 6% in 1987 and oscillates around 4% since the 1990s. Therefore, it seems reasonable to assume that a long-term mean exists and that the interest rate oscillates around this long-term mean. This suggests that the Vasicek and CIR model might be appropriate. The estimation results for the Vasicek and CIR models are presented in Table 5.

Table 5: Estimation results using three-month US treasury constant maturity rates

	α	β	σ	κ	μ
ML estimation for the Vasicek model	0.010475132	-0.239268	0.01257835	0.239268	0.04287734
Approximated Gaussian estimation for the CIR model	0.004617486	-0.1351216	0.04673768	0.1351216	0.03417282
Exact Gaussian estimation for the CIR model	0.005257795	-0.1768168	0.04673768	0.1768168	0.02973582

The estimates for κ and μ are obtained by: $\kappa = -\beta$, $\mu = \alpha / \beta$. The exact Gaussian estimation method produces estimate of σ which is similar to that of Nowman's, but leads to larger estimates of α and β. The long-term mean and the speed of reversion of the Vasicek model estimated by the maximum likelihood method are 4.29% and 0.239628 respectively. The Nowman method provides an estimate of the unconditional mean of 3.42%, while the exact Gaussian method gives a 2.97% estimate. The value of a used in the exact Gaussian estimation method is 0.0013. By choosing different values for a, different sample sequences are obtained. However, the estimates are quite similar, as shown in Table 6. This observation confirms that the selection of parameter a, in an acceptable range (not too big, not too small), will not have a considerable effect on the estimation results. The last column of Table 6 gives the number of sample data selected out of the 6231 observations. By setting a to 0.0013, the sample size is reduced to one tenth of its original size.

Table 6: The exact Gaussian estimation results with different values of *a*

a	α	β	κ	μ	# samples selected
0.0015	0.00597741	-0.19542	0.19542	0.030587	456
0.0014	0.00513331	-0.17503	0.17503	0.029328	488
0.0013	0.00525780	-0.17682	0.17682	0.029736	530
0.0011	0.00515702	-0.17294	0.17294	0.029820	633
0.00087	0.00558760	-0.18343	0.18343	0.030462	777
0.000655	0.00530892	-0.17846	0.17846	0.029748	1015
0.00044	0.00515327	-0.17187	0.17187	0.029984	1464
0.00022	0.00529800	-0.17295	0.17295	0.030631	2621

Figures 3 and 4 show the sample selection resulting from the time transformation of the exact Gaussian method. The grey areas at the bottom of the graphs represent the interest rate, while the black vertical lines indicate the sample selected.

Figure 3: Time transformations for the 3-month US rates (04/01/1982 to 31/12/1985) with *a* = 0.0013

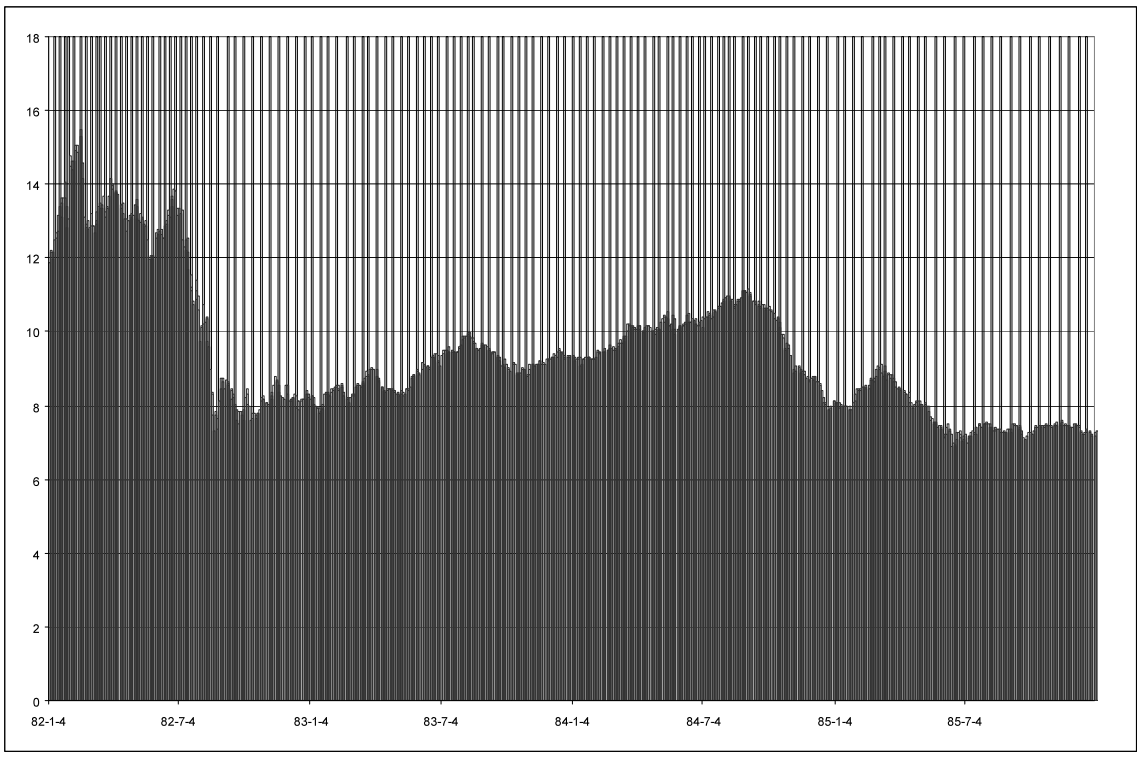
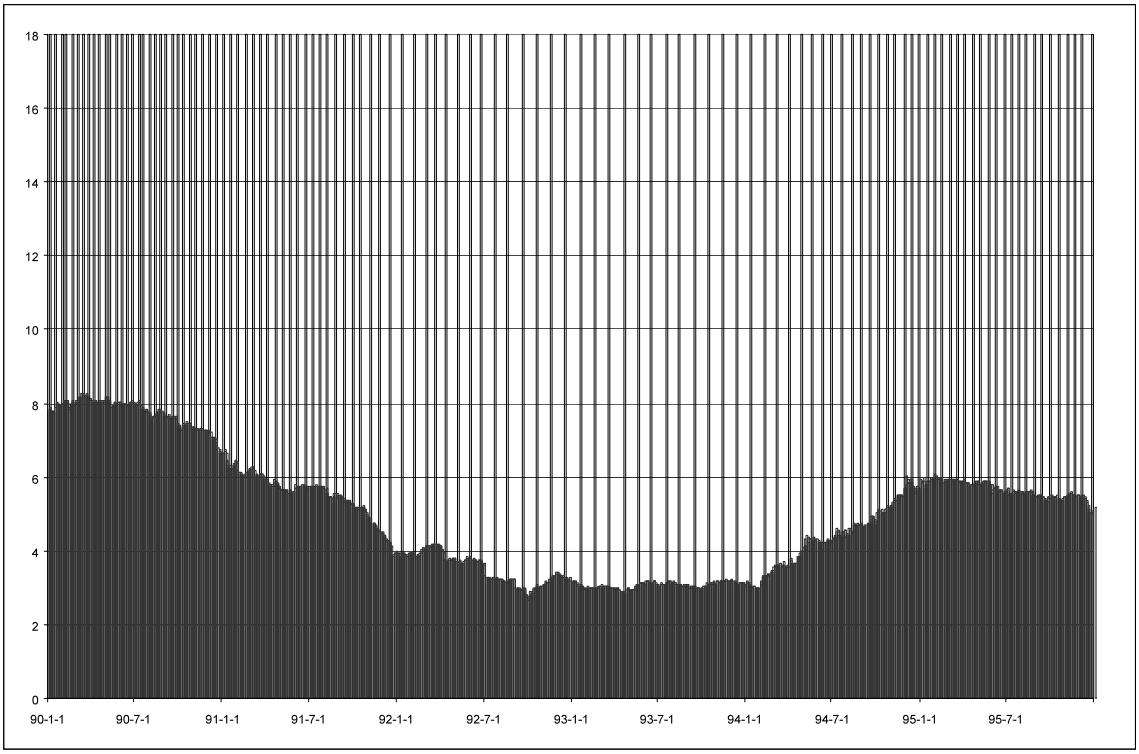


Figure 4: Time transformations for the 3-month US rates (01/01/1990 to 31/12/1995) with *a* = 0.0013



From the graphs, it can be easily observed that the sample selection frequency is higher in high interest rate regions.

Once the estimates of α, β and σ are obtained, the parameter λ is calculated using the least square error method on cross-sectional data. The results are summarized in Table 7. The estimates in Table 7 will be used in the actuarial and financial valuations of the GAOs, which are presented in the following chapters.

Table 7: Estimation results for the Vasicek and CIR models

	κ	μ	σ	λ
Vasicek model	0.239268	0.04287734	0.01257835	-0.58025
CIR model (Exact Gaussian)	0.1768168	0.02973582	0.04673768	-0.12269

4 ACTUARIAL VALUATION OF GUARANTEED ANNUITY OPTIONS

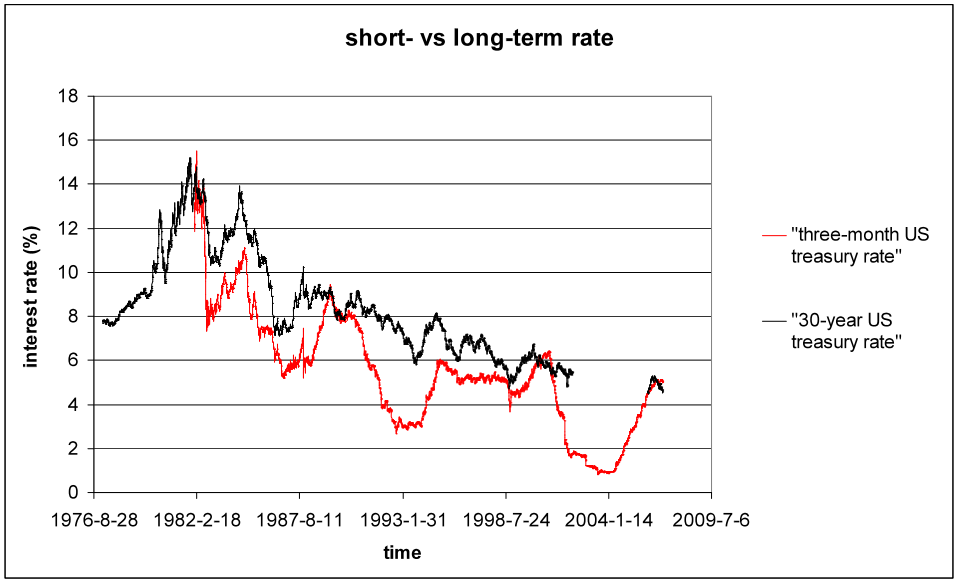
Before going into the actuarial valuation of the guaranteed annuity options, the historical cost of the GAOs are investigated.

4.1 Historical cost of GAOs

The cost of the GAO at maturity for a male aged 65 is formulated as equations (2.1) and (2.2) in Chapter 2. With the assumptions of no expense and fully diversified mortality, the cost of a GAO can be calculated once the mortality table is selected and the interest rates at the time of maturity are known. To find the historical cost of GAOs, the 30-year US treasury constant maturity yield is used. This is the longest rate available on the market. The reason for using the 30-year rate instead of different rates for different maturities is that not all the rates are observable on the market. As the annuities are long-term liabilities to the insurance company, the use of constant 30-year treasury rates seems appropriate.

Figure 5 gives the time series of 3-month and 30-year US treasury constant maturity rates. Note the gap between 18/02/2002 and 09/02/2006 on the 30-year US rate which is due to a lack of data. In this project, the 3-month US treasury rate is used as a proxy of the short-rate $r(t)$.

Figure 5: Time series of short-term and long-term US interest rates



From the figure, it can be observed that the term structure of the interest rate is quite complicated. Most of the time, the long-term rate is above the short-term rate. However, at some points in time, e.g. January 1982, March 1989 and October 2000, the long-term interest rate is below the short-term interest rate. When the long-term rate outperforms the short-term rate, the level of superiority is quite different from time to time. For example, the short-term interest rate in 2006 increases to about the same level as in 1997. However, the long-term interest rate in 2006 is about 5%, which is much lower than the 7% of year 1997. Therefore, it is quite natural for one to question whether the one-factor Vasicek and CIR models are capable of modelling such a complicated term structure. Discussions on this can be found in sections 4.2 and 4.3.

Four different mortality tables are considered when studying the cost of the guarantee:

(1) 1971 Group Annuity Mortality sex-distinct table (GAM71). It was developed specifically for use in the valuation of pension plans before the GAM83 tables were introduced in August 1983. Life expectancy under GAM71 for a male aged 65 is 14.6 years.

(2) 1983 Group Annuity Mortality Table (GAM 83). GAM83 is based on group annuitant experience from 1964 to 1968. GAM83 is probably the most common mortality table used by pension actuaries; 75% of the plans in a 2003 Watson Wyatt survey of actuarial assumptions and funding used GAM83 for funding calculations. Under GAM83, the life expectancy for a male aged 65 is 16.2 years.

(3) The 1994 Uninsured Pensioner Mortality Table (UP94). The UP94 table is based on uninsured pensioner experience projected to 1994. It was developed based on a study of 1985 to 1989 mortality experiences of 29 retirement systems. UP94 is one of the first mortality table to factor in generational mortality, which recognizes the trend of mortality improvement and dynamically projects and incorporates those improvements. Under UP94, the life expectancy for a male aged 65 is 16.76 years.

(4) The Retired Pensioners Mortality Tale (RP2000). The RP2000 table is based on the mortality experience from 1990 to 1994, which is then projected to 2000. It is the only table whose underlying rates are based solely on retirement plan mortality

experience. It was developed by the SOA specifically for current liability calculations. Under RP2000, the life expectancy for a male aged 65 is 17.1 years

The survival probabilities of a male aged 65 under the above four mortality tables are graphed in Figure 6.

Figure 6: Survival probabilities of a male aged 65 using four different mortality tables

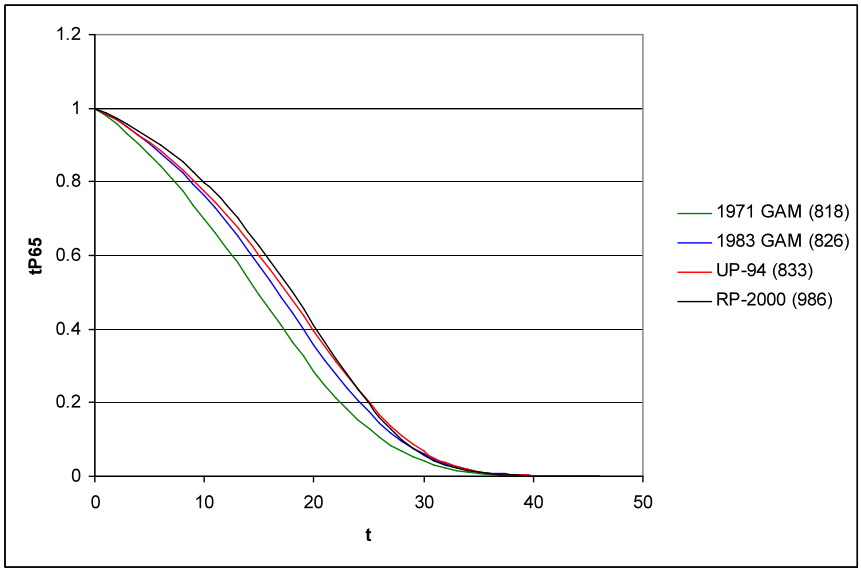
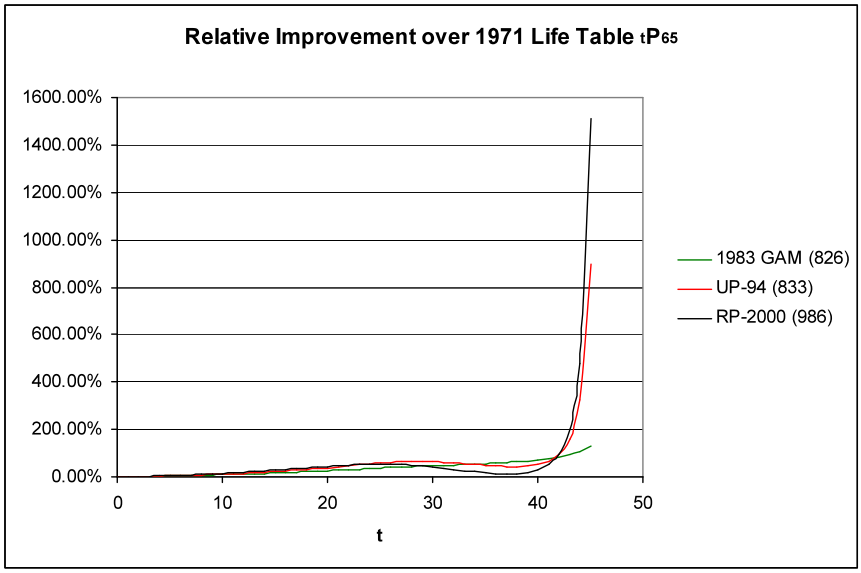


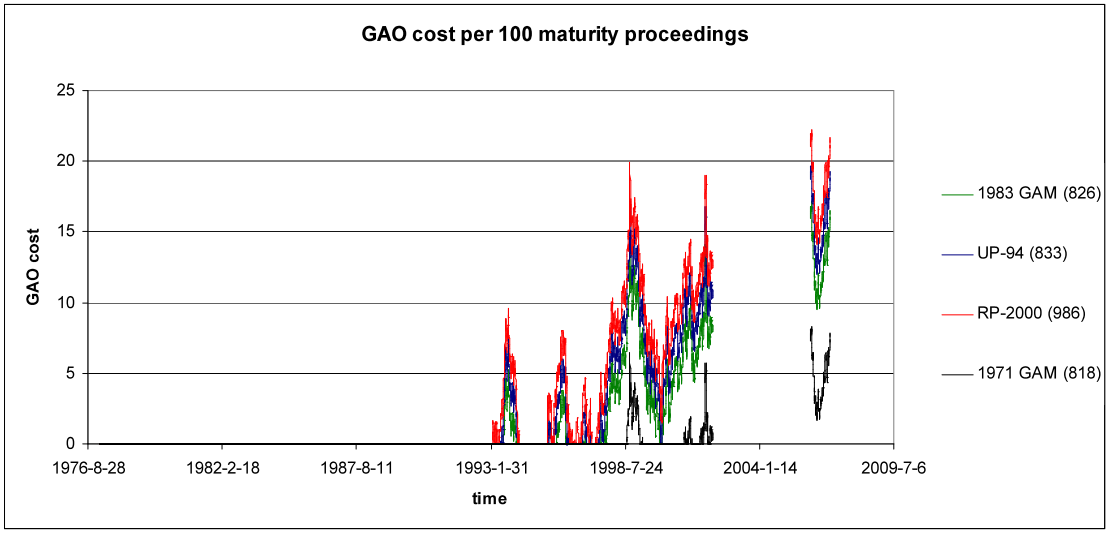
Figure 7 gives the relative improvement (in percentage) of GAM83, UP94, and RP2000 over the GAM71 table. From the graphs, it can be easily observed that there have been substantial improvements in male mortality since the publication of the GAM71 table. The increase in longevity is quite dramatic over the period covered by these four tables. The expectations of life for a male aged 65 are 14.6, 16.2, 16.76, and 17.1 years using GAM71, GAM83, UP94, and RP2000 respectively. Thus the expected future lifetime of a male aged 65 increased by 2.5 years from the GAM71 table to the RP2000 table.

Figure 7: Mortality improvements



As a result of the mortality improvement and the impact of falling long-term interest rate (as shown in Figure 5), the cost of the GAO at maturity increased significantly over the last decade. The evolution of the emerging liability under the GAO is graphed in Figure 8 using four different mortality tables and the historical interest rates. It must be noted that at this stage no option pricing formula or stochastic analysis is involved when calculating the cost of the GAO at maturity. The policy proceeds at maturity are assumed to be held constant at $100 and the cost reported is thus the cost as a percentage of the policy maturity cash value. By applying equation (2.2), with *S(T)* fixed at 100, $a_{65}(T)$ is calculated using the appropriate mortality table and the long-term interest rate series (shown in Figure 5). In addition, the life annuity is assumed to be paid annually in arrear.

Figure 8: Historical cost of GAO per $100 maturity proceedings



From the graph, it can be observed that there is no liability on maturing contracts until the late 1990's when the GAM71 table is used. When other mortality tables are used, however, the liability increases substantially. Under the GAM71, the break-even interest rate for a life annuity is 5.6%. That is, a lump sum of $1000 will purchase an annuity of $111 at an interest rate of 5.6% p.a. The guarantee will be in-the-money if the long-term interest rate is less than 5.6%. Under the GAM83 table, the break-even interest rate is around 6.53%; under the UP94 table, the break-even interest rate is 6.8%, and under the UP2000 table, the break-even interest rate is about 7.04%.

4.2 Cost of GAO using the estimated Vasicek model

In this section, the estimated Vasicek model (as described in Section 3.3) together with the historical short-term interest rates are used to calculate the cost of the GAO at maturity.

From the discussion in Section 3.3, one knows that the maximum likelihood method produces an upward-biased estimate for parameter κ. Therefore, it is quite possible that the estimates we derived in Section 3.6 are not the true values of the model. To test the validity of the ML estimates, a comparison is made between the historical long-term interest rates and the long-tem rates which are calculated by using the estimated Vasicek model and the historical short-term interest rates.

As the ML estimate of parameter κ is usually upward biased, the value of κ is adjusted downward by certain percentages, and the results are examined. Once an acceptable value for parameter κ is found, the corresponding long-term rates are then used to compute the cost of GAOs for a typical contract under four different mortality assumptions. It must be noted that the ML estimated values for the parameters μ and σ are kept unchanged since they proved to be quite accurate in our simulation results (see Section 3.3).

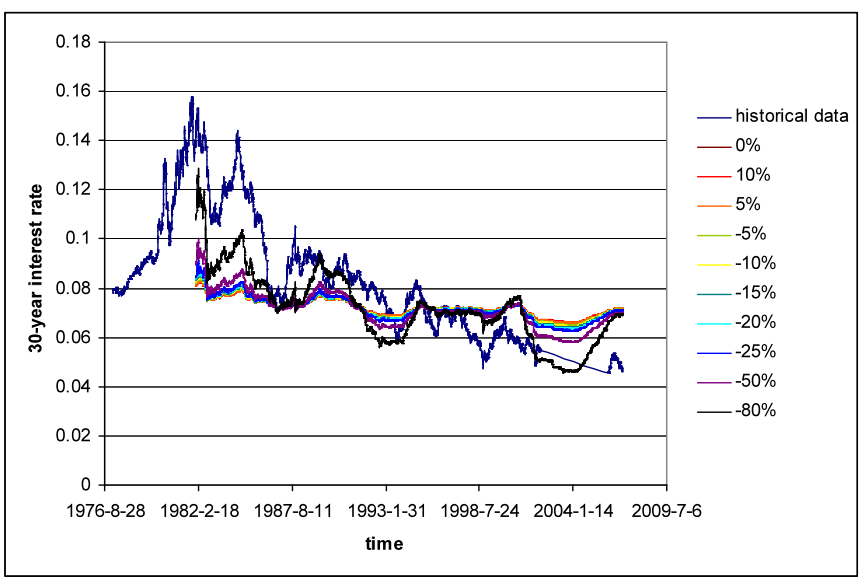
Different values of parameter λ are calculated by using the least square method for different κ, as given in Table 8. The percentages in the first column indicate by how much the value of κ is adjusted (upward or downward) on the basis of its ML estimate.

Table 8: LSE estimates of λ for different values of κ (with μ = 0.04287734 and σ= 0.01257835)

	κ	λ
The ML estimate	0.239268	-0.58025
10%	0.263195	-0.63107
5%	0.251231	-0.60559
-5%	0.227305	-0.55506
-10%	0.215341	-0.53007
-15%	0.203378	-0.50528
-20%	0.191414	-0.48074
-25%	0.179451	-0.4565
-50%	0.119634	-0.34195
-80%	0.047854	-0.23891

The calculated long-term interest rate series using the estimated Vasicek model (with different κ) and the historical short-term interest rates are shown in Figure 9.

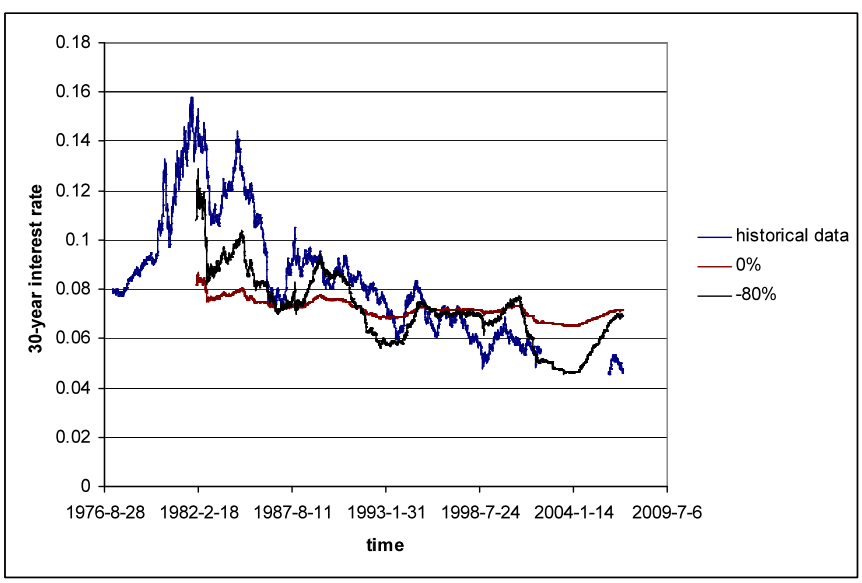
Figure 9: Long-term interest rates calculated by using the historical short-term interest rates and the Vasicek model



The historical long-term interest rate is drawn in blue. The long-term rate calculated using the ML estimate of κ is drawn in brown. Compared to the historical rates, the rates using ML estimate of κ is smoother. As κ decreases, the long-term rates become more volatile and give a better match of the historical data.

From equation (3.1.7), it can be derived that as κ approaches 0, $B(t, T)$ approaches $(T\text{-}t)$ and $lnA(t,T)$ tends to a value which is determined by $(T\text{-}t)$ and σ. Therefore, with $(T\text{-}t)$ =30 and σ fixed, the 30-year interest rate tends to $r(t)$ times a constant when κ is approaching 0. From Figure 5, it can be observed that the movements of the short-term and long-term interest rates are quite consistent most of the time. Long-term interest rate usually moves in the same direction as the short-term rate. As a result, it is not surprising that the long-term rates generated by smaller values of κ better match the historical data, as shown in Figure 10.

Figure 10: Long-term interest rate comparison



However, the mean-reversion speed is determined by κ, and a model with small κ behaves like a random walk process. This obviously contradicts the empirical observation that a long-term mean exists for the interest rate. Therefore, κ cannot be too small. In this project, κ is set to 0.047854 and the corresponding λ is -0.23891. Under such a parameter setting, the calculated cost of the GAO has a similar magnitude as that of its historical cost, as shown in Figure 11.

Figure 11: Cost of GAO per $100 maturity proceedings calculated by using the Vasicek model with adjusted parameters and the historical short-term rates

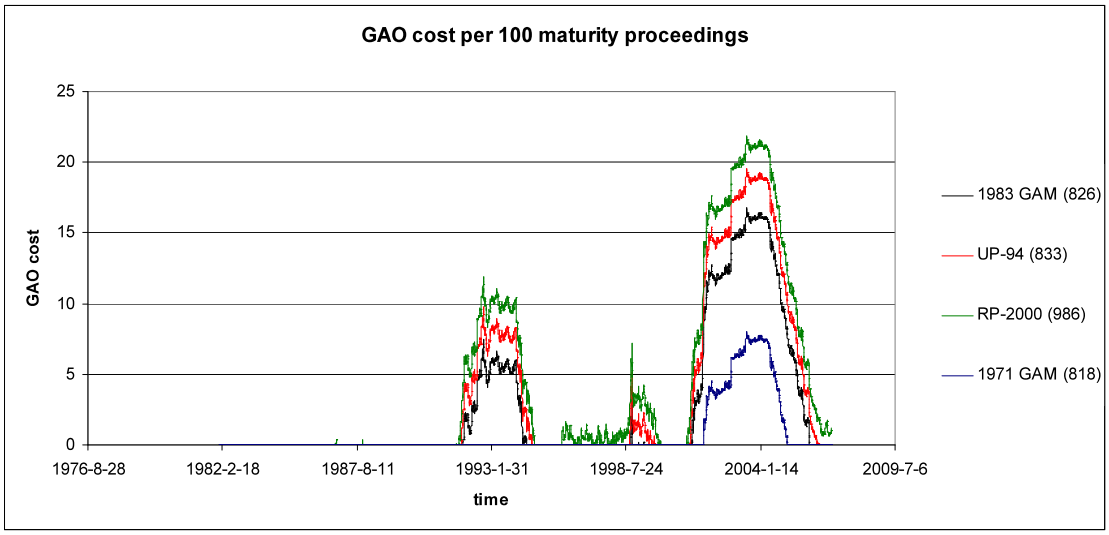
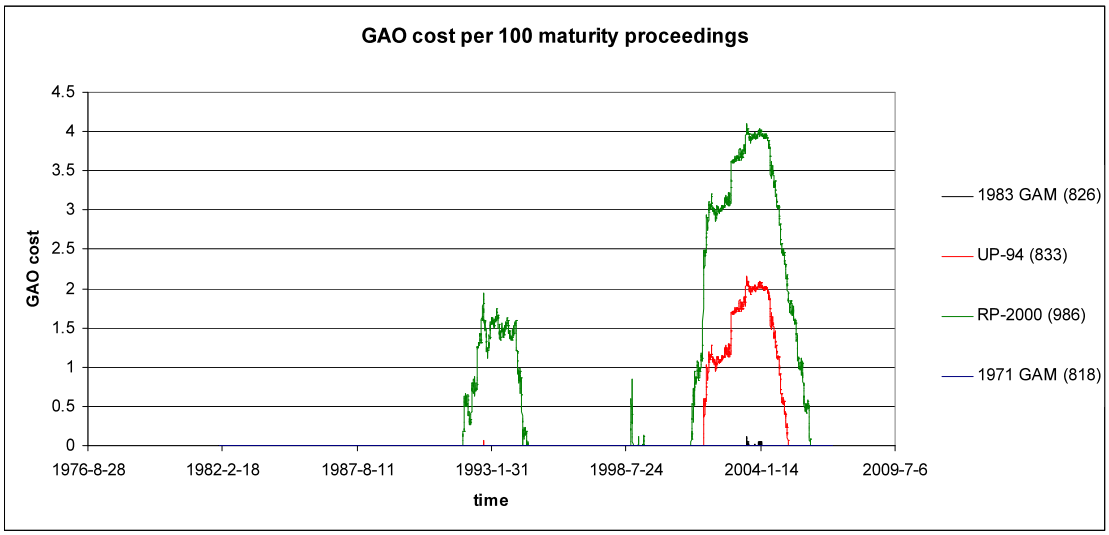


Figure 12 gives the costs of GAO which are calculated using the ML estimates of the Vasicek model. The liabilities are much lower than the historical ones. With mortality assumptions 1971 GAM and 1983 GAM, the calculated costs of GAO are zero under most of the situations.

Figure 12: Cost of GAO per $100 maturity proceedings calculated by using the Vasicek model with ML estimates and the historical short-term rates



4.3 Cost of GAOs using the estimated CIR models

In this section, a similar analysis is performed with the CIR model. The GAO cost calculated by using the CIR model (estimated by the exact Gaussian method) and the historical short-term rate is graphed in Figure 13.

Comparing Figure 13 and Figure 8, one can notice that the cost calculated by the CIR model has a similar shape as the historical cost but at a smaller scale. Trial and error shows that with κ being 0.132613 and λ being -0.10054, reasonable long-term interest rates are obtained (as shown in Figure 14) and hence GAO costs that are comparable to the historical ones can be derived (as shown in Figure 15).

Figure 13: The GAO cost per $100 maturity proceedings calculated by using the CIR model with exact Gaussian estimates and the historical short-rates

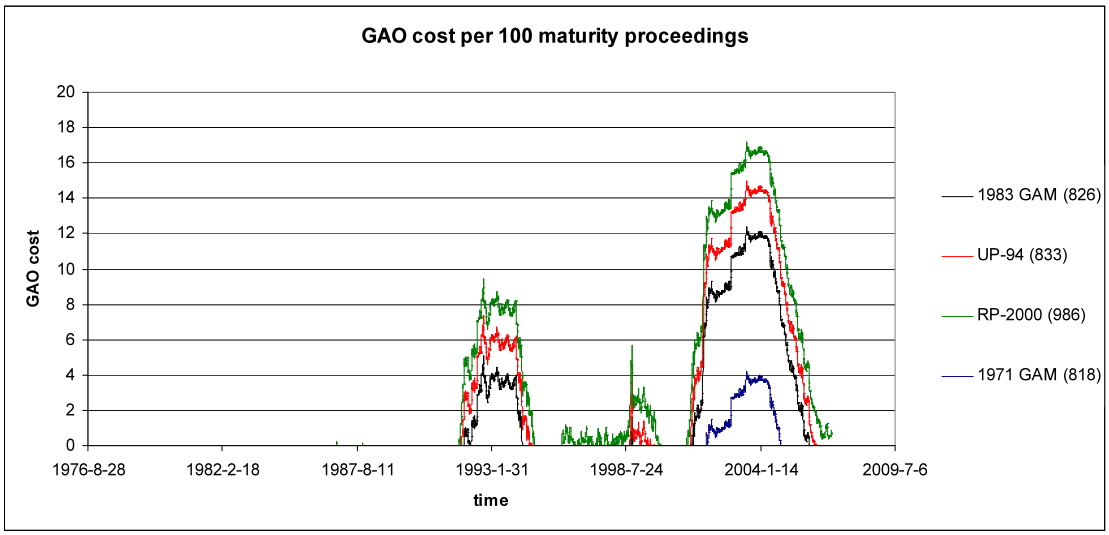


Figure 14: Long-term interest rates calculated using the CIR model and the historical short-rates

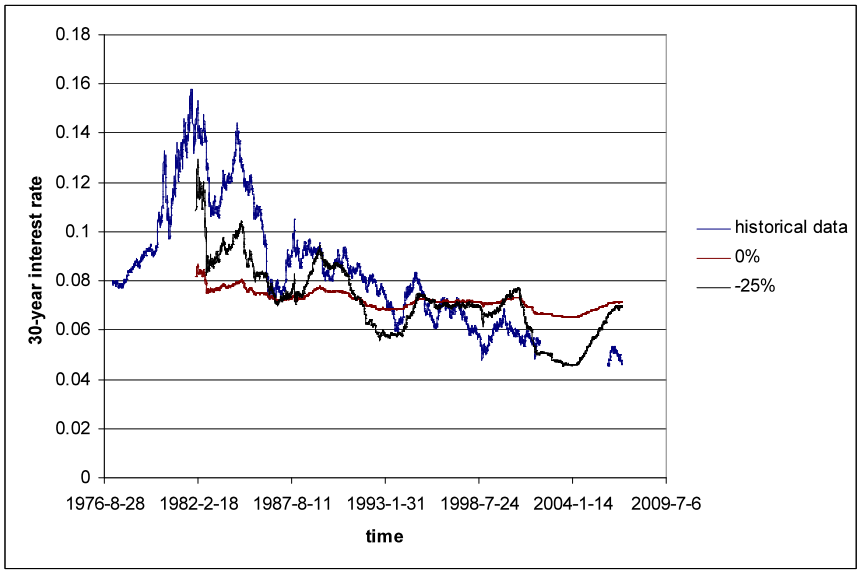
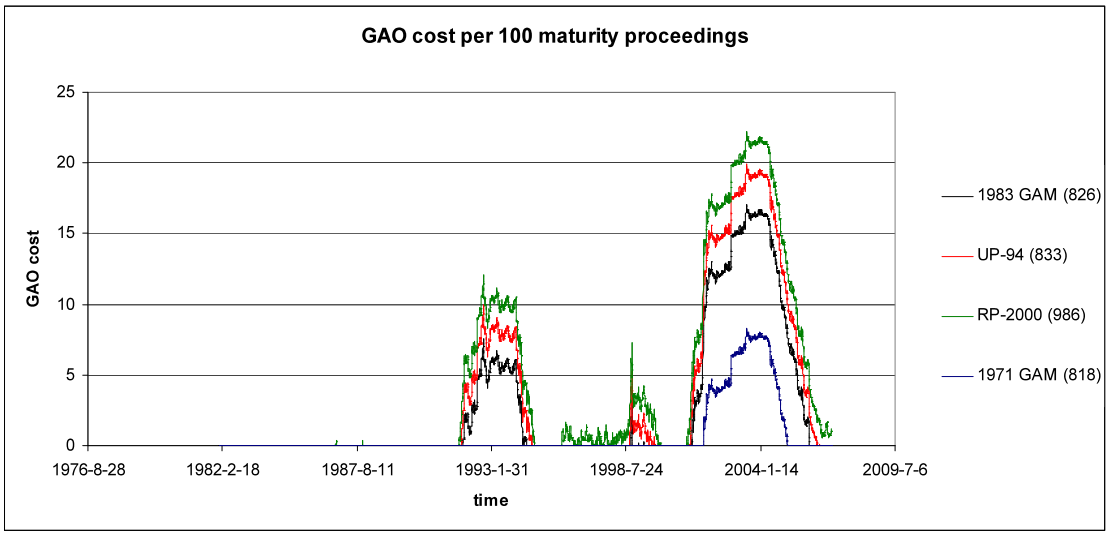


Figure 15: The cost of GAO per $100 maturity proceedings using the CIR model with adjusted parameters and the historical short-rates



4.4 Actuarial valuation

Assume a male aged 65-*T* purchased a GAO contract at time 0. At maturity time *T*, the value of the GAO is $V(T) \quad = \quad {}_T p_{65-T} \times S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right)$ per initial life as discussed in Chapter 2. The valuation problem of GAOs thus becomes to find the discounted value of *V(T)* at time 0, *V(0)*. From equation (2.3), it can be observed that if a premium of ${}_T p_{65-T} \times S(0) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right)$ is collected at time 0 and 100% of the premium is invested in a share portfolio which has market value *S(t)* at time *t*, then the accumulated value of the investment at maturity (or time *T*) will be

$$ {}_T p_{65-T} \times S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right) $$

which is exactly the same as the maturity cost of the GAO per initial life. Therefore, the value of the GAO when the contract is issued at time 0 is

$$V(0) \quad = \quad {}_{T}p_{65-T} \times S(0) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right).$$

At time 0, g is given, $S(0)$ is the market value of the share portfolio and is known. With the assumptions of fully diversified mortality and no mortality improvement over the lifetimes of the pensioners, the survival probability ${}_{t}p_{x}$ can be considered as fixed. From equation (2.2), the value of $a_{65}(T)$ is then determined by the market values of the unit par default-free zero-coupon bonds at time T. However, the time T value of a zero-coupon bond is unknown at time 0. It has a distribution that depends on the movement of interest rates between time 0 and T. Therefore, at time 0, $a_{65}(T)$ is a random variable with a complicated distribution which is determined by the interest rate dynamics.

Like in Wilkie's paper, simulation technique is employed to find the distribution of $V(0)$. Given $r(0)$, the short-rate at time 0, 10,000 values of $r(T)$ are simulated by using the Vasicek or CIR model estimated in the previous sections. The market prices of the unit par zero-coupon bonds are calculated by equation (3.1.7) and the value of $a_{65}(T)$ is then derived for each simulation. Consequently, 10,000 values of $V(0)$ are obtained. The distribution of $V(0)$ can be approximated by the histogram of these $V(0)$ values. Sorting the value of $V(0)$ into an increasing sequence, so that $V(0)_{j-1} < V(0)_j$, the percentiles of $V(0)$ are estimated by the corresponding $V(0)$ values. As in Wilkie et al. (2003), the 99th percentile is $V(0)_{9901}$ and the 99.9th percentile is $V(0)_{9991}$. If the 99th percentile is chosen

as the desired reserve, there is a chance of 1 in 100 that the reserve will be insufficient to cover the actual cost.

Table 9: Present value and quantiles of cost of GAO per $100 single premium: Vasicek model, mortality RP2000, r(0) = 5%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	11.616	30.085	38.198	45.659	55.109	62.097	74.878
15	13.572	36.071	45.522	55.247	65.518	75.979	99.085
20	14.364	38.170	49.497	60.098	72.619	83.247	106.340
25	15.786	43.098	55.591	66.484	80.431	93.184	119.151
30	16.746	45.510	58.007	69.709	85.907	99.983	134.263
35	17.511	48.506	62.252	73.979	91.535	103.912	136.769
40	18.149	49.143	63.477	78.014	98.675	111.661	145.011

Simulation results are shown in Tables 9 to 14. Table 9 is based on the Vasicek model estimated in the previous sections. That is, the present value of cost of GAO is calculated with initial conditions as at the end of December 2006. The initial short-rate $r(0)$ is 5%, and μ = 4.2877%, κ = 0.047854, σ = 1.258%, λ = -0.23891 in the Vasicek model. The mortality table used is RP2000. Policy terms of 10, 15, 20, 25, 30, 35, and 40 are assumed. The quantiles for 90%, 95%, 97.5%, 99%, 99.5%, and 99.9%, are denoted by Q_{90}, Q_{95}, $Q_{97.5}$, Q_{99}, $Q_{99.5}$, and $Q_{99.9}$ respectively in the table.

Table 10: Present value and quantiles of cost of GAO per $100 single premium: CIR model, mortality RP2000, r(0) = 5%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	22.836	36.291	39.208	41.291	43.328	44.677	46.854
15	24.786	38.108	40.644	42.561	44.482	45.394	46.688
20	25.575	38.473	41.048	42.933	44.833	45.949	47.366
25	26.297	39.255	41.761	43.541	45.125	46.044	47.560
30	26.547	39.322	41.566	43.205	45.197	46.048	47.955
35	27.034	39.826	42.430	44.260	45.846	46.735	49.090
40	27.299	40.376	42.752	44.627	46.250	47.179	48.902

Table 10 is based on the CIR model estimated in the previous sections. The initial short-rate rate *r(0)* is 5%, and μ = 2.974%, κ = 0.132613, σ = 4.674%, λ = -0.10054 in the CIR model. From Tables 9 and 10, it can be observed that the discounted present values of the cost of GAOs are not negligible with an initial interest rate of 5%. In addition, the quantiles are quite substantial. There are cases where the quantiles are even higher than the $100 single premium. The mean values of the present value of GAO costs are generally higher when the CIR model is used. A possible reason is that the CIR model has a lower long-term mean than the Vasicek model. The quantiles, however, are another story. All the quantiles obtained by the Vasicek model, except Q_{90}, Q_{95} with T =10 and Q_{90} with T =15 and 20, are larger than those obtained by the CIR model. The higher the quantile, the bigger is the difference. This indicates that the estimated Vasicek model is more volatile than the estimated CIR model.

From the figures, it can also be observed that as the term to maturity increases, the cost of the GAO becomes higher. However, this is not always true. When the initial interest rate is high the cost increases with the term; when the initial interest rate is low, the cost may reduce with the term, or reduce first and increase later, as shown in Tables 11 and 12. When the initial interest rate r(0) is low, there is a greater chance that the interest rate in the short term will also be low, and thus the discounted present value of the cost of GAO will be higher. Comparing Table 9 with Table 11 and Table 10 with Table 12, one can notice that the GAO cost for shorter terms varies much more than that of longer terms when the initial interest rate changes.

Table 11: Present value and quantiles of cost of GAO per $100 single premium: Vasicek model, mortality RP2000, r(0) = 2%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	22.640	46.819	56.426	66.049	76.518	84.405	102.468
15	22.162	49.999	60.590	69.845	82.921	94.151	115.953
20	21.187	50.505	61.676	74.453	88.911	97.951	118.237
25	20.608	50.723	63.806	76.002	91.375	102.677	130.136
30	20.321	51.462	65.456	77.921	94.544	108.357	137.301
35	20.841	54.097	69.069	83.790	100.261	114.777	144.632
40	20.405	53.669	68.978	84.621	101.490	116.875	147.089

Table 12: Present value and quantiles of cost of GAO per $100 single premium: CIR model, mortality RP2000, r(0) = 2%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	28.126	39.913	42.061	43.654	45.072	46.001	47.165
15	27.519	39.580	41.905	43.535	44.896	45.799	47.207
20	27.076	39.267	41.559	43.268	45.060	46.079	47.204
25	26.942	39.473	41.743	43.696	45.027	46.035	47.540
30	27.018	39.876	42.168	44.044	45.759	46.617	47.986
35	27.381	40.174	42.597	44.226	46.010	46.912	48.587
40	27.287	40.344	42.754	44.444	46.189	47.383	48.966

The quantiles calculated above are also commonly known as "Value at Risk' or VaR in its abbreviated form. VaR has, however, been criticised for being 'incoherent'. It is possible for a high quantile (e.g. 70th or 90th) to be smaller than the mean value of a risk. This is unsatisfactory. Another problem is that, when risks are combined into a portfolio, it is possible for the quantile for the portfolio to be greater than the sum of the corresponding quantiles for the individual risks (Wilkie et al. (2003)).

To solve this problem, 'conditional tail expectation' (CTE) can be used. As the αth quantile Q_α of a risk X is defined as $\Pr(X < Q_\alpha) = \alpha\%$, the CTE at level α (denoted by T_α) is defined as $T_\alpha = \mathrm{E}[X \mid X \geq Q_\alpha]$. It is easily calculated during the simulations. For the

sorted 10,000 simulation results of $V(0)$, the CTE at level 99% equals the average of the 100 largest values of $V(0)$, from $V(0)_{9901}$ to $V(0)_{10000}$ inclusive. The CTEs of the cost of a GAO by using the Vasicek and CIR model are presented in Tables 13 and 14 respectively.

Table 13: Present value and CTEs of cost of GAO per $100 single premium: Vasicek model, mortality RP2000, r(0) = 5%

Term to Maturity	Mean	T_{90}	T_{95}	$T_{97.5}$	T_{99}	$T_{99.5}$	$T_{99.9}$
10	11.616	41.155	48.592	55.493	64.166	70.009	82.380
15	13.572	49.534	58.701	67.667	80.367	91.421	111.304
20	14.364	53.702	64.089	74.130	86.780	96.358	115.220
25	15.786	59.869	71.164	82.100	96.652	107.575	129.057
30	16.746	63.321	75.611	87.668	104.705	117.588	145.949
35	17.511	67.207	79.744	92.022	108.976	121.893	153.085
40	18.149	70.077	84.725	99.332	117.942	131.589	169.430

Table 14: Present value and CTEs of cost of GAO per $100 single premium: CIR model, mortality RP2000, r(0) = 5%

Term to Maturity	Mean	T_{90}	T_{95}	$T_{97.5}$	T_{99}	$T_{99.5}$	$T_{99.9}$
10	22.836	39.722	41.764	43.312	44.905	45.931	47.418
15	24.786	41.117	42.928	44.299	45.567	46.202	47.380
20	25.575	41.493	43.303	44.715	46.072	46.752	47.821
25	26.297	42.134	43.829	45.057	46.273	46.963	48.531
30	26.547	42.013	43.641	44.984	46.355	47.113	48.401
35	27.034	42.798	44.544	45.790	47.112	48.005	49.440
40	27.299	43.216	44.921	46.184	47.425	48.145	49.379

From the tables, it can be observed that the CTE values are always greater than the corresponding quantiles, since $T_\alpha = \mathrm{E}[X \mid X \geq Q_\alpha] \geq Q_\alpha$. Besides, the value of any CTE is larger than the mean, since Q_0 is itself equal to the mean, and $Q_\alpha > Q_\beta$ if $\alpha > \beta$. It is also shown (Artzner, 1998) that, when risks are combined into a portfolio, the portfolio CTE cannot be greater than the sum of the individual CTEs.

Once the distribution of $V(0)$ is found, various premium calculating principles and reserving techniques can be applied. Pricing and reserving for the GAO are not discussed in this project.

5 FINANCIAL PRICING AND HEDGING OF GUARANTEED ANNUITY OPTIONS

The similarities between the guaranteed annuity options and other types of financial option have been pointed out by many researchers. Among them are: Wilkie et al. (2003), Boyle & Hardy (2003), Bolton et al. (1997), and Pelsser (2002). The work presented in the project is mainly based on Boyle & Hardy (2003), in which modern option pricing and dynamic hedging techniques are used.

5.1 Option pricing

The value of a GAO at maturity time T is $S(T)\times\max\left(a_{65}(T)/g-1,\quad 0\right)$ where $a_{65}(T)=\sum_{n=1}^{\omega-65}{}_{n}p_{65}\times D_{T+n}(T)$ (as given by equations (2.1) and (2.2)). From the formula, it is quite obvious that a GAO is similar to a call option on a coupon bond with the annuity payments and survival probabilities being incorporated in the notional coupons.

The price at maturity time T of a zero-coupon bond with unit maturity value, maturing at $T+n$ ($n \geq 1$) is denoted as $D_{T+n}(T)$ or $P(T,T+n)$. The value of $D_{T+n}(T)$ at time T depends on the term structure which is assumed to be known at time T. At time $t < T$, however, $D_{T+n}(T)$ is a random variable driven by a stochastic interest rate model. By the no-arbitrage theorem described in Chapter 4 of Cairns (2004), the value of a risk at time t ($t < T$) which has payoff $V(T)$ at time T is

$$V(t) = P(t,T)E_t^{Q_T}[V(T) \mid F_t] \tag{5.1.1}$$

where Q_T is a new probability measure (distribution), the so-called *forward-risk* adjusted measure. This technique was introduced in the fixed-income literature by Jamshidian (1991). As a result, under the fully diversified mortality assumption, the value of a GAO at time 0 is

$$V(0) = {}_T p_{65-T} \times P(0,T) \times E^{Q_T}\left[S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0\right)\right]. \tag{5.1.2}$$

The market value of the share portfolio $S(T)$ is also a random variable and is assumed to be independent of interest rates. Note this may be a very strong assumption but it simplifies the analysis. The value of the GAO at time 0 thus becomes:

$$\begin{aligned} V(0) &= \frac{{}_T p_{65-T}}{g} P(0,T)E^{Q_T}[S(T)]E^{Q_T}[(a_{65}(T) - g)^+] \\ &= \frac{{}_T p_{65-T}}{g} S(0)E^{Q_T}\left[(a_{65}(T) - g)^+\right]. \end{aligned} \tag{5.1.3}$$

The last line follows since $S(0) = P(0,T)E^{Q_T}[S(T)]$ under the no-arbitrage theorem. Inserting the expression for $a_{65}(T)$ from (2.2), we have

$$V(0) = \frac{{}_T p_{65-T}}{g} S(0)E^{Q_T}\left[\left(\sum_{n=1}^{\omega-65} {}_n p_{65} \times P(T, T+n) - g\right)^+\right]. \tag{5.1.4}$$

The expression inside the expectation on the right hand side corresponds to a call option on a coupon paying bond where the 'coupon' payment at time $(T+n)$ is ${}_n p_{65}$ and the expiration date is time T. This 'coupon bond' has value at time T:

$$\sum_{n=1}^{\omega-65} {}_n p_{65} \times P(T, T+n).$$

The market value at time t of this coupon bond is

$$P(t) = \sum_{n=1}^{\omega-65} {}_n p_{65} \times P(t, T+n).$$

So $P(t)$ is the value of a deferred annuity, but without allowance for mortality before retirement. With the notation $P(t)$, equation (5.1.4) becomes

$$V(0) = \frac{{}_T p_{65-T}}{g} S(0) E^{Q_T}[(P(T) - g)^+]. \tag{5.1.5}$$

Jamshidian (1989) showed that if the interest rate follows a one-factor process, then the market price of the option on the coupon bond with strike price g is equal to the price of a portfolio of options on the individual zero-coupon bonds with strike prices K_n, where $\{K_n\}$ are equal to the notional zero-coupon bond prices to give an annuity $a_{65}(T)$ with market price g at T. That is, let r_T^* denote the value of the short-rate at time T for which

$$\sum_{n=1}^{\omega-65} {}_n p_{65} D^*(T, T+n) = g \tag{5.1.6}$$

where the asterisk is used to indicate that each zero-coupon bond is evaluated using the short-rate r_T^*. K_n is then set to $K_n = D^*(T,T+n)$. The call option with strike g and expiration date T on the coupon bond $P(t)$ can be valued as

$$C\left[P(t),g,t\right]=\sum_{n=1}^{\omega-65}{}_{n}p_{65}C\left[P(t,T+n),K_{n},t\right] \tag{5.1.7}$$

where $C\left[P(t,T+n),K_{n},t\right]$ is the price at time t of a call option on the zero-coupon bond with maturity (T+n), strike price K_n and expiration date T. Under the forward-risk measure,

$$C[P(t),g,t]=P(t,T)E^{Q_T}[(P(T)-g)^{+}\mid F_t]. \tag{5.1.8}$$

Thus

$$E^{Q_T}\left[\left(P(T)-g\right)^{+}\mid F_t\right]=\frac{\sum_{n=1}^{\omega-65}{}_{n}p_{65}C\left[P(t,T+n),K_{n},t\right]}{P(t,T)}$$

and by equation (5.1.5), we have

$$V(0)=\frac{{}_{T}p_{65-T}}{g}S(0)\frac{\sum_{n=1}^{\omega-65}{}_{n}p_{65}C\left[P(0,T+n),K_{n},0\right]}{P(0,T)}. \tag{5.1.9}$$

Two interest rate models have been estimated and studied in the previous chapters. They are the Vasicek model and the CIR model. Experimental results show that with proper parameter settings, both models can generate reasonable cost of GAOs. However, as the closed-form solution for the option price is very complicated under the CIR model, only the Vasicek model is considered in this chapter.

Under the Vasicek model (see Section 3.1), the price at time t of a call option on a zero-coupon bond with strike price *g*, maturity date *T*+*n*, and expiration date *T* is given by equation (5.1.10) (see Boyle & Hardy (2003)).

$$C\left[P(t,T+n),K_n,t\right]=P(t,T+n)N(h_1(n,t))-K_nP(t,T)N(h_2(n,t)) \qquad (5.1.10)$$

where

$$\begin{aligned} h_1(n,t) &= \frac{\ln P(t,T+n)-\ln P(t,T)-\ln K_n}{\sigma_P(n,t)}+\frac{\sigma_P(n,t)}{2}, \\ h_2(n,t) &= \frac{\ln P(t,T+n)-\ln P(t,T)-\ln K_n}{\sigma_P(n,t)}-\frac{\sigma_P(n,t)}{2}=h_1(n,t)-\sigma_P(n,t) \end{aligned} \qquad (5.1.11)$$

and

$$\sigma_P(n,t)=\sigma\sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}}\frac{(1-e^{-\kappa n})}{\kappa}. \qquad (5.1.12)$$

Table 15: The cost of GAO at time 0 obtained by option pricing with r(0) = 0.05

	T=10	T=15	T=20	T=25	T=30	T=35	T=40
V(0)	6.585170	6.729368	6.883999	7.052726	7.228360	7.413138	7.613177

The parameters of the Vasicek model are set to: *r(0)* = 5%, μ = 4.2877%, κ = 0.047854, σ = 1.258%, and λ = -0.23891. The prices of the zero-coupon bonds with different maturities can be calculated by using equation (3.1.7). After that, the price of a call option on the zero-coupon bond can be calculated by (5.1.10) and the resultant value

of the GAO at time 0 is obtained by (5.1.9). Table 15 shows, for terms of 10, 15, 20, 25, 30, 35 and 40 years, the initial values of the GAO per $100 single premium, using mortality table RP2000.

Compared with Table 9, it can be observed that the option pricing costs are smaller for all terms than the mean costs calculated using the actuarial method. Under the actuarial method described in Section 4, all the premiums are invested into an equity account. With the option pricing method, however, the premiums are invested into equities and bonds with different maturities according to certain proportions. As a result, the maturity values of GAOs should be discounted back with different rates and hence the $V(0)$ values are different for these two methods as shown in Table 9 and Table 15. Note that in Table 9, the values all increase considerably with the term. In Table 15, the values of longer terms are not much greater than those of shorter terms. Of course with different parameters different results might be obtained.

The results for various initial conditions are given in Table 16. The first row gives the values of $V(0)$ when κ is doubled. Comparing them to Table 15, one can observe that $V(0)$ increases for $T = 10$ but decreases for all the other terms when κ is doubled. The effect of doubling κ is more obvious for longer terms.

The results for doubling the volatility σ are tabulated in the second row. As the interest rates become more volatile, the probability that the insurer has to pay for the GAO becomes larger and the cost of the GAO becomes higher, as shown in Table 16.

The last two rows of the table give the effect of changing the initial interest rate $r(0)$. When $r(0)$ is lower, the chance that $r(t)$ will remain low becomes higher, and the value of $V(0)$ will thus be larger.

Table 16: The cost of GAO at time 0, V(0), calculated with different parameter settings

	T=10	T=15	T=20	T=25	T=30	T=35	T=40
$r(0) = 5\%$, $\kappa = 0.09571$ $\sigma = 1.258\%$ $\lambda = -0.23891$	6.834641	6.624155	6.491909	6.422260	6.398067	6.413455	6.467144
$r(0) = 5\%$, $\kappa = 0.047854$ $\sigma = 2.516\%$ $\lambda = -0.23891$	12.21211	15.34684	18.69096	22.15480	25.60556	28.95980	32.18586
$r(0) = 2\%$, $\kappa = 0.047854$ $\sigma = 1.258\%$ $\lambda = -0.23891$	13.731644	11.760524	10.595791	9.868453	9.401490	9.110786	8.952066
$r(0) = 8\%$, $\kappa = 0.047854$ $\sigma = 1.258\%$ $\lambda = -0.23891$	2.599921	3.487202	4.232036	4.880734	5.451819	5.962125	6.428779

5.2 Dynamic hedging

In order for the above theoretical option price (5.1.9) to be taken as the true or 'fair' value of the GAO, a hedging strategy must exist such that the results of the investments according to the hedging strategy replicate the desired payoff of the GAO. To hedge the GAO against both the equity and interest risks, we would need to invest in the following securities: an equity index with market value $S(t)$ at time t, zero-coupon bond maturing at time T, and zero-coupon bonds maturing at time $T+n$ ($n = 1, \ldots, \omega$-65).

A delta hedging strategy is employed here. The hedging ratios are thus the partial derivatives of $V(t)$ over the corresponding underlying securities. The number of units invested in the index at time t is denoted by $H_S(t)$ and is equal to

$$\begin{aligned} H_s(t) &= \frac{{}_T p_{65-T}}{g} \frac{\sum_{n=1}^{\omega-65} {}_n p_{65} C\left[P(t,T+n), K_n, t\right]}{P(t,T)} \\ &= \frac{{}_T p_{65-T}}{g} \sum_{n=1}^{\omega-65} {}_n p_{65} \left[\frac{P(t,T+n)}{P(t,T)} N(h_1(n,t)) - K_n N(h_2(n,t)) \right] \end{aligned} \tag{5.2.1}$$

where $h_1(n,t)$ and $h_2(n,t)$ are given by equations (5.1.11) and (5.1.12).

The second consists of an investment at time t of $H_0(t)$ units of the zero-coupon bond which matures at time T, where

$$H_0(t) = -\frac{{}_T p_{65-T}}{g} S(t) \sum_{n=1}^{\omega-65} {}_n p_{65} \frac{P(t,T+n)}{P(t,T)^2} N(h_1(n,t)). \tag{5.2.2}$$

The replicating portfolio also consists of investments of $H_n(t)$ units of the zero-coupon bonds which matures at time $T+n$ ($n = 1, \ldots, \omega$-65), where

$$H_n(t) = \frac{{}_T p_{65-T}}{g} S(t) {}_n p_{65} \frac{N(h_1(n,t))}{P(t,T)} \qquad n = 1, \ldots, \omega - 65. \tag{5.2.3}$$

If the limiting age of the policyholder is 110, we have to invest at all times in the 47 securities according to the above hedging proportions. Note that the value of the initial hedge is

$$
\begin{aligned}
& H_s(0)S(0) + H_0(0)P(0,T) + \sum_{n=1}^{\omega-65} H_n(0)P(0,T+n) \\
&= \frac{{}_T p_{65-T}}{g} S(0) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[\frac{P(0,T+n)}{P(0,T)} N(h_1(n,t)) - K_n N(h_2(n,t)) \right] \\
&\quad - \frac{{}_T p_{65-T}}{g} S(0) \sum_{n=1}^{\omega-65} {}_n p_{65} \frac{P(0,T+n)}{P(0,T)} N(h_1(n,t)) \\
&\quad + \frac{{}_T p_{65-T}}{g} S(0) \sum_{n=1}^{\omega-65} {}_n p_{65} \frac{P(0,T+n)}{P(0,T)} N(h_1(n,t)) \\
&= \frac{{}_T p_{65-T}}{g} S(0) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[\frac{P(0,T+n)}{P(0,T)} N(h_1(n,t)) - K_n N(h_2(n,t)) \right] \\
&= H_s(0)S(0) = V(0)
\end{aligned}
$$

which is equal to the value of the GAO at time 0. At maturity time T, $P(T,T) = 1$, $\sigma_P(n,T) = 0$, $h_1(n,T) = h_2(n,T) = +\infty$, $N(h_1(n,T)) = N(h_2(n,T)) = 1$, and hence the value of the hedge portfolio is

$$
\begin{aligned}
& H_s(T)S(T) + H_0(T)P(T,T) + \sum_{n=1}^{\omega-65} H_n(T)P(T,T+n) \\
&= \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[P(T,T+n) - K_n \right] \\
&\quad - \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} P(T,T+n) \\
&\quad + \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} P(T,T+n) \\
&= \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[P(T,T+n) - K_n \right] = V(T).
\end{aligned}
$$

That is, the result of the investment process matches exactly the required payoff of the GAO at maturity time T. Suppose the hedge is to be rebalanced at time $t+h$. Just before rebalancing, the value of the hedge portfolio is

$$H_s(t)S(t+h) + H_0(t)P(t+h,T) + \sum_{n=1}^{\omega-65} H_n(t)P(t+h,T+n)$$

where $S(t+h)$, P($t+h$,T), and P($t+h$,T+n) denote the market prices at time $t+h$ of the hedge assets. The new hedging weights $H_S(t+h)$, $H_0(t+h)$, and $H_n(t+h)$ are computed based on these new asset prices and the value of the revised hedge is

$$G(t+h) = H_s(t+h)S(t+h) + H_0(t+h)P(t+h,T) + \sum_{n=1}^{\omega-65} H_n(t+h)P(t+h,T+n)\,.$$

Note, the value of the hedge portfolio at time t is denoted by $G(t)$.

If the 'real world' model is the same as the model used for option pricing, and if hedging is carried out continuously, free of transaction costs, then the hedge portfolio is self-financing since the hedging proportions are actually the partial derivatives of $G(t)$ over the corresponding assets. However, the rebalancing is done discretely in practice. In addition, there are transactions costs and the market movements can deviate significantly from those implied by the model. All these can lead to considerable hedging errors.

From the above discussion, it can be concluded that the value of the replicating portfolio, $G(t)$, will almost certainly not exactly match the value of $V(t)$. The following investment strategy is then considered: invest the correct amounts in the portfolio, and invest the balance if there's any in the zero-coupon bond, or borrow the shortage by shorting the zero-coupon bond. The assumption implied in the above investment strategy is that we can borrow or lend at the risk-free rate, which is of course not true in the real world.

The hedging results of 10,000 simulations with h=1/250 (daily rebalancing) are tabulated in Table 17. The hedging is carried out according to the investment strategy described in the previous paragraphs. $S(t)$ is assumed to follow the stochastic equation $dS(t) = \mu_s S(t)dt + \sigma_s S(t)dW_t$, where $\mu_s = 0.1$, $\sigma_s = 0.2$, and W_t is the standard Brownian motion. Note that $S(t)$ is independent of $r(t)$ by assumption, $S(0)$ is \$100, and $r(0)$ is 5%.

Table 17: Investments in equity and bonds of the replicating portfolio in the 10,000 simulations

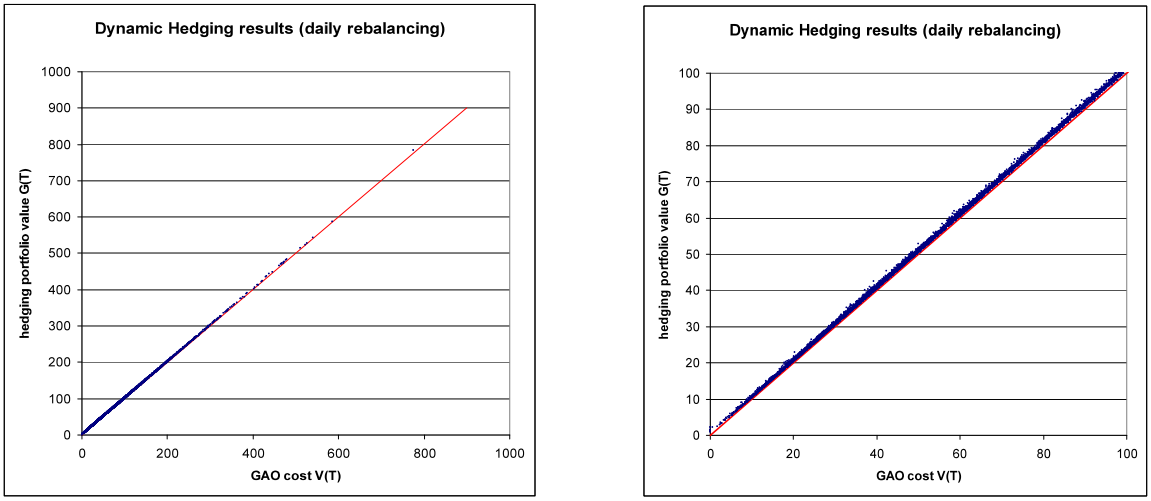
	t=0		t=5		t=10	
	Mean	variance	mean	variance	mean	Variance
$H_s(t)*S(t)$	6.5852	0.0000	16.4778	315.6496	45.4460	5997.6140
$H_0(t)*P(t,T)$	-46.0180	0.0000	-1.0390	6226.1250	-2.4508	76179.8500
$H_1(t)*P(t,T+1)$	4.1620	0.0000	9.2514	44.4455	21.2216	541.0267
$H_2(t)*P(t,T+2)$	3.9349	0.0000	8.7687	40.5846	20.2033	492.2916
$H_3(t)*P(t,T+3)$	3.7051	0.0000	8.2767	36.7417	19.1517	444.7861
$H_4(t)*P(t,T+4)$	3.4748	0.0000	7.7805	32.9813	18.0785	399.0195
$H_5(t)*P(t,T+5)$	3.2458	0.0000	7.2844	29.3561	16.9940	355.3937
$H_6(t)*P(t,T+6)$	3.0197	0.0000	6.7921	25.9075	15.9074	314.2068
$H_7(t)*P(t,T+7)$	2.7978	0.0000	6.3065	22.6644	14.8259	275.6496
$H_8(t)*P(t,T+8)$	2.5810	0.0000	5.8298	19.6455	13.7554	239.8252
$H_9(t)*P(t,T+9)$	2.3700	0.0000	5.3639	16.8631	12.7009	206.7922
$H_{10}(t)*P(t,T+10)$	2.1655	0.0000	4.9106	14.3251	11.6672	176.5880
$H_{11}(t)*P(t,T+11)$	1.9683	0.0000	4.4717	12.0350	10.6593	149.2239
$H_{12}(t)*P(t,T+12)$	1.7790	0.0000	4.0490	9.9929	9.6822	124.6918
$H_{13}(t)*P(t,T+13)$	1.5984	0.0000	3.6441	8.1947	8.7409	102.9488
$H_{14}(t)*P(t,T+14)$	1.4269	0.0000	3.2587	6.6311	7.8393	83.9013
$H_{15}(t)*P(t,T+15)$	1.2650	0.0000	2.8935	5.2887	6.9805	67.4144
$H_{16}(t)*P(t,T+16)$	1.1129	0.0000	2.5495	4.1520	6.1676	53.3316
$H_{17}(t)*P(t,T+17)$	0.9704	0.0000	2.2263	3.2002	5.3999	41.4292
$H_{18}(t)*P(t,T+18)$	0.8379	0.0000	1.9251	2.4176	4.6809	31.5472
$H_{19}(t)*P(t,T+19)$	0.7158	0.0000	1.6468	1.7869	4.0139	23.5047
$H_{20}(t)*P(t,T+20)$	0.6044	0.0000	1.3923	1.2896	3.4016	17.1012
$H_{21}(t)*P(t,T+21)$	0.5038	0.0000	1.1621	0.9069	2.8456	12.1228

	t=0		t=5		t=10	
	Mean	variance	mean	variance	mean	Variance
$H_{22}(t)*P(t,T+22)$	0.4141	0.0000	0.9564	0.6197	2.3469	8.3508
$H_{23}(t)*P(t,T+23)$	0.3351	0.0000	0.7748	0.4102	1.9052	5.5724
$H_{24}(t)*P(t,T+24)$	0.2665	0.0000	0.6169	0.2622	1.5199	3.5896
$H_{25}(t)*P(t,T+25)$	0.2079	0.0000	0.4818	0.1612	1.1892	2.2242
$H_{26}(t)*P(t,T+26)$	0.1588	0.0000	0.3684	0.0950	0.9112	1.3210
$H_{27}(t)*P(t,T+27)$	0.1189	0.0000	0.2760	0.0537	0.6838	0.7525
$H_{28}(t)*P(t,T+28)$	0.0871	0.0000	0.2024	0.0291	0.5022	0.4104
$H_{29}(t)*P(t,T+29)$	0.0624	0.0000	0.1451	0.0151	0.3607	0.2140
$H_{30}(t)*P(t,T+30)$	0.0437	0.0000	0.1017	0.0074	0.2532	0.1065
$H_{31}(t)*P(t,T+31)$	0.0299	0.0000	0.0697	0.0035	0.1737	0.0506
$H_{32}(t)*P(t,T+32)$	0.0200	0.0000	0.0466	0.0016	0.1164	0.0230
$H_{33}(t)*P(t,T+33)$	0.0131	0.0000	0.0305	0.0007	0.0763	0.0100
$H_{34}(t)*P(t,T+34)$	0.0084	0.0000	0.0195	0.0003	0.0489	0.0041
$H_{35}(t)*P(t,T+35)$	0.0052	0.0000	0.0122	0.0001	0.0306	0.0016
$H_{36}(t)*P(t,T+36)$	0.0032	0.0000	0.0075	0.0000	0.0188	0.0006
$H_{37}(t)*P(t,T+37)$	0.0019	0.0000	0.0045	0.0000	0.0113	0.0002
$H_{38}(t)*P(t,T+38)$	0.0011	0.0000	0.0026	0.0000	0.0066	0.0001
$H_{39}(t)*P(t,T+39)$	0.0006	0.0000	0.0015	0.0000	0.0038	0.0000
$H_{40}(t)*P(t,T+40)$	0.0004	0.0000	0.0009	0.0000	0.0022	0.0000
$H_{41}(t)*P(t,T+41)$	0.0002	0.0000	0.0005	0.0000	0.0012	0.0000
$H_{42}(t)*P(t,T+42)$	0.0001	0.0000	0.0003	0.0000	0.0007	0.0000
$H_{43}(t)*P(t,T+43)$	0.0001	0.0000	0.0002	0.0000	0.0004	0.0000
$H_{44}(t)*P(t,T+44)$	0.0000	0.0000	0.0001	0.0000	0.0002	0.0000
$H_{45}(t)*P(t,T+45)$	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000

From Table 17, it can be observed that the amounts invested into equity and bonds are known with certainty at time 0. As t approaches maturity time T = 10, higher variances are exhibited. The summation of the first column equals 6.58, which is exactly the GAO value for T=10 in Table 15. This proves that the value of the replicating portfolio at time 0 is the value of the GAO, $V(0)$.

To investigate whether the value of the replicating portfolio at maturity time T, $G(T)$, matches the payoff $V(T)$ required by the GAO, another 10,000 hedging simulations are performed. The parameters of the interest rate model and the equity model are the same. The initial short-rate r(0) varies between 3% and 20% in these 10,000 simulations. The simulation results are graphed in Figure 16.

Figure 16: Hedging results of 10,000 simulations (daily rebalancing) (a) plots in normal scale; (b) partial view



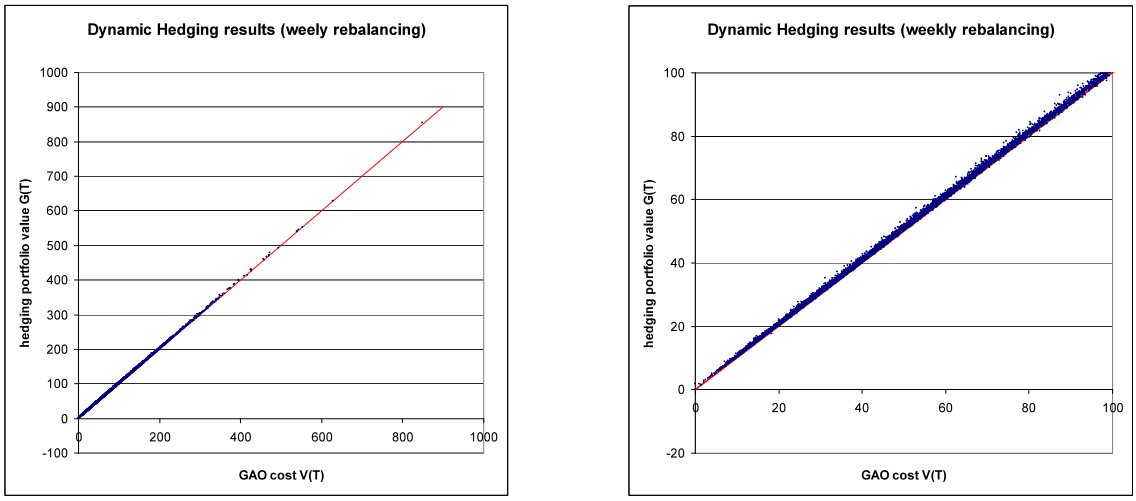
(a) (b)

In Figure 16, the values of $G(T)$, the investment proceeds at maturity time T, are plotted against the values of $V(T)$, the amount required to pay off the option at maturity. One can see that in general the investment proceeds correspond with the amounts required very closely. Surpluses at maturity (although not significant) can be observed when the hedging strategy was followed.

Figures 17 and 18 show the simulation results with the same variables but hedging weekly and monthly respectively. From these figures, it can be observed that

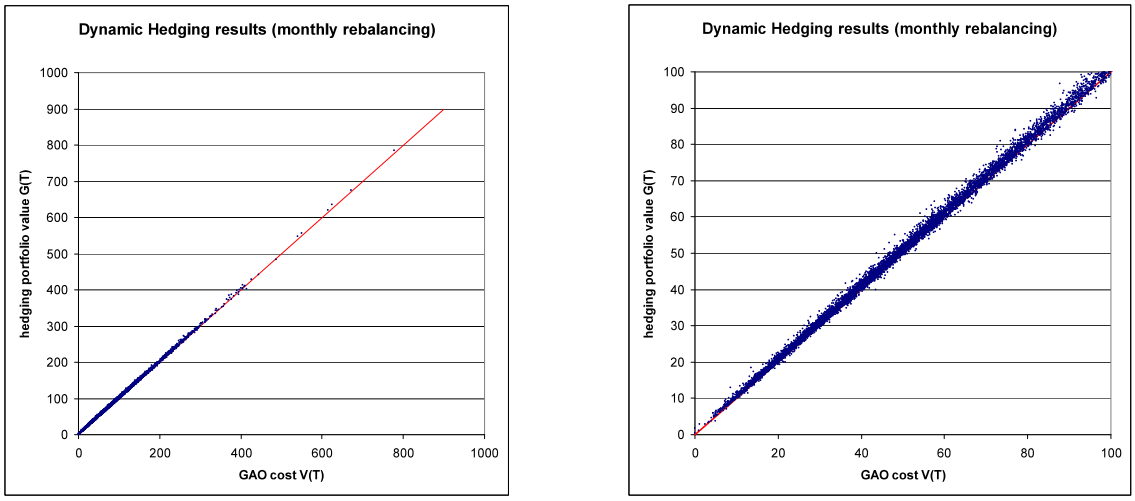
although the investment results cluster around the 45-degree line, the correspondence between the $V(t)$ and $G(T)$ is by no means perfect. Proportionately large profits and deficits can be observed especially with monthly rebalancing.

Figure 17: Hedging results of 10,000 simulations (weekly rebalancing) (a) plots in normal scale; (b) partial view



(a) (b)

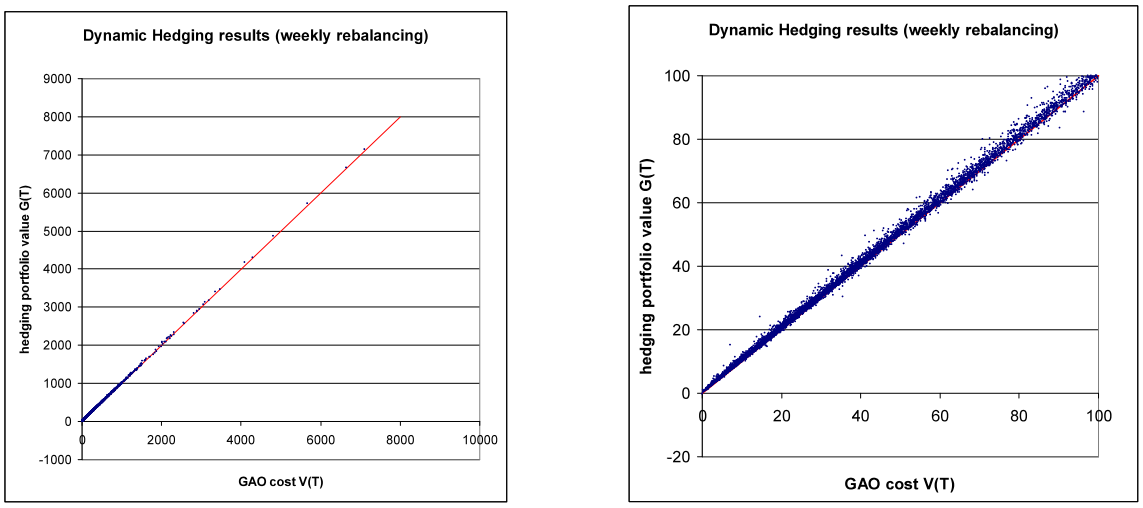
Figure 18: Hedging results of 10,000 simulations (monthly rebalancing) (a) plots in normal scale; (b) partial view



(a) (b)

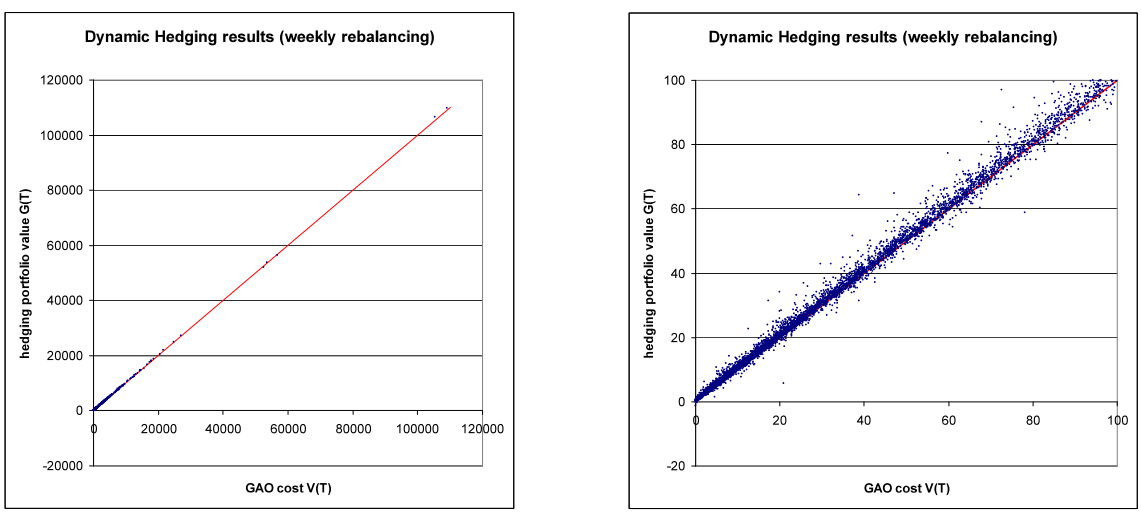
As the results depend strongly on the value of $S(T)$, the effect of changing the volatility σ_s is investigated and presented in Figures 19 and 20.

Figure 19: Hedging results of 10,000 simulations (weekly rebalancing, σ_s = 0.3) (a) plots in normal scale; (b) partial view



(a) (b)

Figure 20: Hedging results of 10,000 simulations (weekly rebalancing, σ_s = 0.4) (a) plots in normal scale; (b) partial view



(a) (b)

One can see that as the $S(t)$ process becomes more volatile, the correspondence between $G(t)$ and $V(t)$ becomes less close. The extreme values are far more extreme when $\sigma_s = 0.4$ than when $\sigma_s = 0.2$.

The results demonstrate that the investment strategy we have described above, if hedging is sufficiently frequent, and if the real-world model is as assumed, does give results that correspond with the required payoff quite closely. This validates that the option and hedging formula (5.1.9) is appropriate for the guaranteed annuity options. In the project, it is assumed that the real-world interest rates in fact behave in accordance with the Vasicek model that has been defined and used for the calculation of option values and hedging quantities. The true behaviour of the real-world interest rates may actually be very different. In addition, $S(t)$ is assumed to have a log-normal distribution while it is now well established in the empirical literature that equity prices do not follow a simple lognormal process (Boyle & Hardy (2003)). All these and many other frictional factors, such as transaction costs which are not considered in the project, will almost certainly widen the hedging errors in real-world applications.

6 CONCLUSIONS

In this work, the value of Guaranteed Annuity Options (GAOs) is investigated in an environment of stochastic interest rates. The maturity value of a GAO at time T is modelled by a mathematical model and its discounted value at time t is calculated under the following assumptions: 1) fully diversified mortality; 2) no mortality improvement; 3) no expense; 4) US market; 5) mortality risk independent of the financial risk and 6) uncorrelated equities and bonds.

Two methods are employed to find the discounted value of GAO at time t. They are the so-called actuarial method and the financial pricing method. With the actuarial method, all premiums are invested into an equity account and the discounted value of GAO at time 0 can be modelled by an option on the deferred annuity $a_{65}(T)$. To find the value of $a_{65}(T)$, two one-factor stochastic interest rate models are introduced: the Vasicek model and the CIR model. Methods for their estimation are described and tested with simulated data. The experiment results suggest that the risk-neutral long-term mean and the volatility can be estimated quite accurately while the estimate of the mean-reversion rate is upward biased. The estimated Vasicek and CIR models are then calibrated using cross sectional data. The maturity costs of the GAOs are calculated using the calibrated models and are compared against the costs calculated using the historical long-term interest rates. Experimental results show that these calibrated models can give

reasonable results and thus can be used to find the discounted value of the GAO at time 0, $V(0)$. Means, variances, percentiles, and conditional tail expectations of $V(0)$, for different maturities, are found by Monte Carlo simulation.

The second method is the financial pricing method. Under this method, the value of $V(0)$ can be calculated using the option pricing formulas. Only the Vasicek model is employed as it is difficult to find a closed-form option price solution for the CIR model. The values of V(0) for different maturities are calculated and presented. A replicating portfolio is constructed and a delta hedging strategy is employed to make sure that the results of the investments replicate the desired payoff of the GAO. The experiment results demonstrate that, if rebalancing is sufficiently frequent, the investment strategy matches with the required payoff quite closely.

This work shows that both the actuarial and financial methods are able to give reasonable valuations of the GAOs under proper assumptions. However, the results obtained by these two methods are not the same as $V(0)$ is invested and discounted differently. Therefore, a possible direction for future work would be to analyze the relationships between these two sets of results. Delta hedging strategy is employed in this work. When the volatility is high, however, other hedging strategies such as delta-gamma hedging might be more appropriate. Moreover, no mortality improvement is considered in this work. This is obviously not the case in real world. The valuation models can thus be made more realistic by including mortality improvements. Finally, a more challenging problem would be to study GAOs when the mortality risk is not fully diversified.

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