ANALYSIS OF JOINT LIFE INSURANCE WITH STOCHASTIC INTEREST RATES

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Abstract

A general portfolio of joint life insurance contracts is studied in a stochastic interest rate environment with independent and dependent mortality models. Two types of joint insurance products, namely joint first-to-die and joint last-to-die, are considered in this project. Two methods are used to derive the first two moments of the prospective loss random variable. The first one is based on the individual loss random variables while the second one studies annual stochastic cash flows. The total riskiness of the portfolio is decomposed into its insurance risk and its investment risk. For illustrative purposes, an AR(1) process is used to model the stochastic interest rates. Copula and common shock models are used to model dependence in mortality. The effects of mortality dependence on the riskiness of portfolios of joint life insurances are analyzed.

Keywords: Joint Life Insurance, Stochastic Rate of Return, AR(1) Process, Dependent Mortality Model, Frank's Copula, Common Shock Model

To my parents

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Chapter 1

Introduction

Insurance organizations offer contractual promises to provide a contingent payment on the death of an individual (death benefit), in return for a series of periodical payments (premium). These life insurance contracts, are designed to protect against serious financial impact that results from an individual's death. An important variation of the "standard" (single) life insurance is the so called joint life insurance where multiple lives are involved. Under these contracts, death benefits are payable on whichever death within a group. For instance, a joint-life insurance issued to a married couple pays the death benefit if either spouse dies.

Basically, there are two key components in the valuation of joint life insurance; one is the time value of money and the other is the policyholders' survivorship.

The time value of money is an important component for valuations because premiums and benefit payments are made in the future. The classic interest rate model where the interest rate is deterministic which neglects the possible variation of the future interest rates. Since 1970's, many stochastic interest rate models has been proposed. The main advantage of stochastic interest rate models is the capability to capture the likelihood and magnitude of random deviations from the long term mean. The conditional Autoregressive process of order one, AR(1) was applied Panjer and Bellhouse (1980) for applications in life contingencies, while the Moving Average model, MA(1), was used by Frees (1990) to address the stochastic theory of valuation from a solvency perspective. The more general Autoregressive Integrated Moving Average Model, ARIMA(p,d,q), was proposed by Dhaene (1989). The author illustrated that this higher-order model can also be used for the short-term rates and may capture more information about the variation of the term structure. A homogeneous time-continuous Markov Chain model was proposed by Norberg (1995) and the numerical solutions were obtained for expected values of a wide class of functionals of the process, including moments of present values of payment streams. Other interest rate models can be found in Cairns (2004). In our project, we assume a conditional AR(1) model which is frequently used in the actuarial literature. With this assumption for interest rates, it is possible to derive formulas for the moments of the discounted value and calculate them numerically.

The policyholders' survivorship is also an important component, since the payments are made when death occurs. We now review some mortality models for single life in the actuarial literature. For individual lives, De Moivre's Law was introduced in 1729 as the first law of mortality where deaths are assumed to happen uniformly over the interval of deaths. Then the 2-parameter Gompertz Law was proposed and became a popular one. There were generalizations of Gompertz Law, for example, 3-parameter Makeham, 7-parameter Thiele, 4-parameter Perks, 2-parameter Weibull, 5-parameter Beard and 4-parameter Barnette. Forfar (2006) summarized the above mortality laws and examined how mortality might be improved in the foreseeable future. We limit our discussion to the Gompertz distribution for individual lives.

To extend the survivorship from individuals to pairs, an assumption on the dependence between the survivorship of each individual is necessary. The assumption of independence between lives is commonly used in the early years when valuing joint life insurance. The main advantage of this assumption is the simplicity of the calculation. With this assumption, the joint survival probability can be expressed as the product of individual survival probabilities. Then the problem of the joint survivorship is reduced to a single-life survivorship problem. However, several empirical studies regarding joint lives have shown the evidence of the dependence of lives. There are several ways to model the impact of survivorship of one life upon another. A traditional one is the common shock model, introduced for dependent mortality modeling, in which a common variable (or shock) is included in each individual mortality structure. This common random variable usually is used to model some additional hazards from natural catastrophes, such as an earthquake or an aircraft crash, which affect the individual's mortality risk. As a result, the individual's future lifetime might be shortened by this common shock. The main advantages of using the common shock model are that the model can be interpreted easily and computed conveniently. Copula, as a popular parametric joint survival model, applied in Genest and McKay (1986), can be used to construct bivariate distributions with given marginal distributions in a pair. In the actuarial literature, many possible copula functions have been presented. For example, Gaussian copula was introduced by Li (2000) for financial modeling, and the T copula, is a similar one to Gaussian copula except that its dependent structure is implicitly described by a multivariate t distribution. A one-parameter copula family in the Archimedean copulas family due to Frank (1979) and named Frank's copula was commonly used to model the joint life distribution in actuarial practice. A detailed review of the wide range of copulas and their applications can be found in Nelsen (2006) . For illustrative purposes, common shock and Frank's copula models are used to model dependence in mortality of a pair in this project.

The aim of this project is to extend the theory on valuation of single life insurance to valuation of a general (non-homogeneous) joint life insurance portfolio with stochastic interest rate assumption and dependent mortality assumption. Our focus is the first two moments of the prospective loss random variables, since they can provide us with an insight into the properties of the life insurance products and help the insurance company determine a contingency reserve. Following Parker (1997) and Marceau and Gaillardetz (1999), two approaches are used to calculate the first two moments of the prospective loss random variable for the whole portfolio under our model setting. The first one bases on the individual loss random variable while the second one studies annual stochastic cash flows. These two approaches provide us with two different views of the loss random variable for the portfolio. The first approach studies each contract in the portfolio which is easy to understand. The second approach focuses on the payouts of the portfolio which is useful for the approximation of the distribution of the loss random variable for the portfolio. The total riskiness of the portfolio is decomposed into its insurance risk and investment risk.

The rest of the project is organized as follows. Chapter 2 provides a brief literature review of joint life insurance valuation. The main contributions and results of the key papers on various life insurance valuation models in the actuarial literature are presented. Chapter 3 describes the detailed assumptions regarding the decrements and discounting functions that will be used in later chapters. In Chapter 4, two expressions are given for the valuation of prospective loss random variable for a general joint-life insurance portfolio, while analogous formulas for a general last-survivor life insurance portfolio valuation are developed in Chapter 5. In Chapter 6, numerical illustrations are shown for a homogeneous portfolio with an AR(1) stochastic interest rate assumption and two mortality models.

Chapter 7 concludes the project.

Chapter 2

Literature Review

In this chapter, we review some results from papers which made important contributions on life insurance models. There is an extensive literature on this topic, and hence we focus on the models and their actuarial application instead of a complete list of related papers.

The theory of life contingencies evolves from single life policy valuations. Deterministic treatment of interest rates is a traditional approach in the theory of life contingencies where the time until decrement (death) is considered as the only random variable.

A lognormal model was used in Boyle (1976) by assuming that the force of interest follows

$$\delta(k) - \delta = \varepsilon_k \quad k = 1, 2, \cdots , \tag{2.1}$$

where δ is the long term mean of the force of interest, $\delta(k)$ is a random variable representing the force of interest in year k and $\{\varepsilon_k; k = 1, 2 \cdots\}$ is a sequence of independent normally distributed random variables with mean zero. The author used this assumption to calculate the first three moments of actuarial functions. With the same interest rate assumption, Waters (1978) further derived the first four moments of the summation of individual actuarial functions. The drawback of the lognormal interest rate assumption above is the independent assumption on the force of interest rate in any other year.

Panjer and Bellhouse (1980) further developed a general theory including continuous and discrete stochastic interest rate models. The lognormal model given in (2.1) and the stationary autoregressive processes of order one, AR(1) given in (2.2) were specifically considered in the derivation of the moments of the present value of an annuity certain and a life annuity. If a sequence of $\{\delta(k); k = 1, 2, \cdots\}$ is an AR(1) model centered around δ, then,

$$\delta(k) - \delta = \phi[\delta(k-1) - \delta] + \varepsilon_k \quad k = 1, 2, \cdots , \tag{2.2}$$

where $\{\varepsilon_k; k = 1, 2, \cdots\}$ is a sequence of i.i.d. random variables following a normal distribution with mean zero and finite variance σ^2, i.e., $\varepsilon_k \sim N(0, \sigma^2)$, δ is the long-term mean of the process, and ϕ is the coefficient of the process. The assumption that $|\phi| < 1$ is made in order for the model to remain stationary. The main contribution of their paper is the derivation of the actuarial functions with autoregressive model (2.2). Lysenko (2006) examined the behavior of insurance surplus over time for a portfolio of homogenous life policies with an AR(1) model for rate of return.

A Moving Average model of order one for the force of interest, MA(1) was considered in Frees (1990) to address the stochastic theory of valuation from a solvency perspective. A sequence of $\{\delta(k); k = 1, 2, \cdots\}$ is said to follow an MA(1) model centered around δ if

$$\delta(k) - \delta = \varepsilon_k - \theta\varepsilon_{k-1} \quad k = 1, 2, \cdots ,$$

where $\{\varepsilon_k; k = 1, 2, \cdots\}$ is a sequence of i.i.d. random variables following a normal distribution with mean zero and finite variance σ^2, i.e., $\varepsilon_k \sim N(0, \sigma^2)$ and δ is the long-term mean of the process. With $\theta = 0$, the model reduces to lognormal model described in (2.1). The author also examined the performance of the loss random variable for a portfolio under a correlated MA(1) interest environment. Frees (1997) used the coefficient of determination which qualifies the relative importance of a single source to all risks under a MA(1) interest rate model assumption.

Dhaene (1989) further considered the Autoregressive Integrated Moving Average process, ARIMA(p,d,q), as a model for interest rates, and computed the present value functions in an efficient manner. A sequence of $\{\delta(k); k = 1, 2, \cdots\}$ is said to follow an ARIMA(p,d,q) model if

$$\nabla^d\delta(k) = a + b_1\nabla^d\delta(k-1) + \cdots + b_p\nabla^d\delta(k-p) + \varepsilon_t - c_1\varepsilon_{t-1} - \cdots - c_q\varepsilon_{t-q} ,$$

where ∇^d stands for the d^{th} backward difference operator:

$$\nabla^1\delta(k) \equiv \nabla\delta(k) = \delta(k) - \delta(k-1),$$

$$\nabla^d\delta(k) \equiv \nabla(\nabla^{d-1}\delta(k)) , \quad d = 2, 3, \cdots ,$$

and

$$a = \delta \left(1 - \sum_{i=1}^{p} b_i \right) .$$

$b_1, \cdots, b_p$ are the parameters of the autoregressive part of the model, the $c_1, \cdots, c_q$ are the parameters of the moving average part of the model, δ is the long-term mean of the process, and $\{\varepsilon_k; k = 1, 2, \cdots\}$ are error terms. The error terms $\{\varepsilon_k; k = 1, 2, \cdots\}$ are generally assumed to be independent, identically distributed variables from a normal distribution with zero mean. The author demonstrated that the higher-order ARIMA process can be used to model the short-term rates and may capture more information about the variation of the term structure.

Lai and Frees (1995) considered both the traditional linear ARIMA model and the newer nonlinear Autoregressive Conditionally Heteroskedastic, ARCH(p,q), process to model the force of interest. A sequence of $\{\delta(k); k = 1, 2, \cdots\}$ is said to follow an ARCH model if

$$\delta(k) = a + b_1\delta(k-1) + \cdots + b_p\delta(k-q) + \varepsilon_t - c_1\varepsilon_{t-1} - \cdots - c_q\varepsilon_{t-q} ,$$

with

$$\varepsilon_t|\Psi_{t-1} \sim N(0, h_t) ,$$

$$h_t = \varphi_0 + \varphi_1\varepsilon_{t-1}^2 ,$$

where Ψ_t is the information set (σ-field) available at time t-1, $\varphi_0 > 0$, $\varphi_1 > 0$. The coefficients $b_1, \cdots, b_p$, $c_1, \cdots, c_q$ and the terms $\{\varepsilon_k; k = 1, 2, \cdots\}$ have the same meaning as in the ARIMA model. Under this nonlinear time series setting, the conditional variance of one-step-ahead prediction is time variant. The main advantage of the ARCH model is its ability to capture the tendency of volatility clustering, that is, for large changes to be followed by large changes and small changes to be followed by small changes. The author stated that an explicit expression for the reserve can be obtained under a linear interest rate process while reserves under a nonlinear interest rate process can be calculated by approximation formulas and simulation algorithms.

Continuous-time stochastic interest rate models also play an important role in life insurance valuation. The most popular one is the Ornstein-Uhlenbeck process, also called the Vasicek model, for which the instantaneous interest rate at time k, $\delta(k)$, satisfies the following first order Stochastic Differential Equation (SDE).

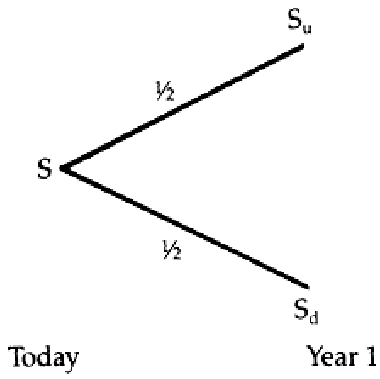
$$d\delta(k) = -\alpha(\delta(k) - \delta)dt + \sigma dW_t ,$$

where δ is the long term mean, α is the speed of reversion which describes the velocity of convergence to the long term mean, σ measures the instantaneous volatility and W_t is the standard Brownian motion. Pandit and Wu (1983) introduced a principle of covariance equivalence to set up a discrete representation of the continuous system, for instance, AR(1) is the discrete representation of Ornstein-Uhlenbeck process. By assuming that the force of interest accumulation function follows an Ornstein-Uhlenbeck process, Beekman and Fuelling (1991) obtained explicit expressions for the moments of an annuity certain and a life annuity. Parker (1994) compared the modeling of the force of interest and the modeling of force of interest accumulation function by using Ornstein-Uhlenbeck processes. The first three moments of the present value of a life annuity under these two models were presented in the paper. As pointed out by the author, when modeling the force of interest accumulation function, $y(t)$, the conditional expected value of $y(t)$ given $y(s)$ and $\delta(s)$, $s < t$, is independent of the value of the force of interest at time s, $\delta(s)$. Parker (1998) presented some methods to obtain the distributions of actuarial functions which are useful approaches for insurance portfolio pricing or valuation.

By assuming that the force of interest follows an Ornstein-Uhlenbeck process, Parker (1997) presented two approaches to calculate the moments of the loss random variable of a general portfolio which contains different types of insurance contracts. The first one is based on the individual loss random variable while the second one studied annual stochastic cash flows. The author provided an insight into the total riskiness of the whole portfolio by splitting the variance into two parts, corresponding to the so-called insurance risk and investment risk. This paper proposed a method to approximate the loss random variable in a limiting case where the number of contracts in a portfolio is large. Marceau and Gaillardetz (1999) further investigated the properties of the loss random variable for a portfolio with AR(1) modelling the force of interest. This paper allows contracts to have different issue dates by including an indicator function in the formulas. Following Parker (1997), Bruno et al. (2000) allocated the total riskiness of the portfolio to management periods and studied the risk due to interactions.

Black (1990) first introduced the Black-Derman-Toy (BDT) binomial model for short-term rates. In the BDT model, an interest-rate-sensitive security, worth S today, either goes up to S_u or down to S_d over a year (see Figure 2.1). These S, S_u and S_d can be used to determine the implicit interest rate. The detailed analysis of the BDT model is discussed by Panjer et al. (1998) and Lin (2006). Gaillardetz (2007) presented an approach to price

Figure 2.1: A one-step tree



variable annuity financial guarantees with a BDT model.

The deterministic mortality assumption has been used as a traditional approach in insurance valuations. When a joint life insurance is considered, the independent assumption for lives of individuals in a group is often made to simplify the calculations, though it is not realistic to ignore completely the dependency in individual's mortality. Frees et al. (1996) investigated the use of dependent mortality models for determining annuity values. A copula model was proposed as a survivorship function for joint lives and the common shock model was also discussed. Further more, data from a large insurance company were used to model and examine these dependent mortality models and calculate annuity values with constant interest rates.

This project investigates the performance of a portfolio of joint life insurance contracts with a stochastic model for interest rates and a dependent mortality model for pairs. The parameters in the dependent mortality model are obtained from Frees et al. (1996). Detailed assumptions regarding the model for the force of interest and decrements will be presented in Chapter 3.

Chapter 3

Model Assumptions

3.1 Stochastic Interest Rate

Stochastic processes have been introduced to model the interest rate since the 1970's. The main advantage of stochastic interest rate models is their capability to capture the likelihood and magnitude of random deviations from the long term mean. As it has been seen from the literature review in Chapter 2, the conditional autoregressive process of order one, AR(1), is a simple discrete-time stochastic model for interest rates. In this project, we assume an AR(1) interest rate model for our illustration.

Let $\delta(k)$ be the force of interest over the k^{th} year, $(k-1, k]$, for $k = 1, 2, \cdots$. The AR(1) model for sequence $\{\delta(k); k = 1, 2, \cdots\}$ is defined by

$$\delta(k) - \delta = \phi[\delta(k-1) - \delta] + \epsilon_k \ , \tag{3.1}$$

where $\{\epsilon_k;\ k = 1, 2, \cdots\}$ is a sequence of i.i.d. random variables following a normal distribution with mean zero and finite variance σ^2, i.e., $\epsilon_k \sim N(0, \sigma^2)$, δ is the long-term mean of the process, and ϕ is the coefficient of the process. The assumption that $|\phi| < 1$ is made in order for the model to remain stationary. The stationary property of the process can ensure the existence of its first and second moments as time t tends to infinity. Under this model, the unconditional moments (expectation, variance and covariance) of the force of interest can be easily obtained. In fact, for $k = 1, 2, \cdots$,

$$E[\delta(k)] = \delta \ , \tag{3.2}$$

$$Var[\delta(k)] = \frac{\sigma^2}{1-\phi^2} \ , \tag{3.3}$$

$$Cov[\delta(k), \delta(k+m)] = \frac{\sigma^2}{1-\phi^2}\phi^m \ , \quad m = 0, 1, 2 \cdots \ . \tag{3.4}$$

The so-called discount factor, which discounts future cash flows to current time can be expressed as

$$v(k) = e^{-I(k)} \quad k = 1, 2, \cdots \ ,$$

where $I(k) = \sum_{j=1}^{k} \delta(j)$ is called the force of interest accumulation function for k periods. Its unconditional moments can be derived based on the unconditional moments of the force of interest. We list these results as a proposition below.

Proposition 1. *Assume that* $\{\delta(j); j = 1, 2, \cdots\}$ *follows the AR(1) model given by* (3.1)*. Then the* 1^{st} *and* 2^{nd} *order unconditional moments of* $\{I(k); k = 1, 2, \cdots\}$ *are given by*

$$E[I(k)] = k\delta \ ,$$

$$Var[I(k)] = \frac{\sigma^2}{1-\phi^2}\left[k + 2\frac{\phi}{1-\phi}\left(k - 1 - \frac{\phi}{1-\phi}\left(1-\phi^{k-1}\right)\right)\right] \ , \tag{3.5}$$

and when $s < k$*,*

$$Cov[I(s), I(k)] = Var[I(s)] + \frac{\sigma^2}{1-\phi^2}\frac{\phi}{(1-\phi)^2}\left(\phi^s - \phi^k\right)\left(\phi^{-s} - 1\right) \ ,$$

where $Var[I(s)]$ *is given by Eq.* (3.5)*.*

Proof. By using (3.2)-(3.4), we have

$$E[I(k)] = E\left[\sum_{j=1}^{k} \delta(j)\right] = \sum_{j=1}^{k} E\left[\delta(j)\right] = k\delta \ ,$$

$$\begin{aligned} Var[I(k)] &= Var\left[\sum_{j=1}^{k} \delta(j)\right] = \sum_{j=1}^{k}\sum_{i=1}^{k} Cov[\delta(j), \delta(i)] \\ &= kVar[\delta(j)] + 2\sum_{j=1}^{k-1}\sum_{i=j+1}^{k} Cov[\delta(j), \delta(i)] \\ &= k\frac{\sigma^2}{1-\phi^2} + 2\sum_{j=1}^{k-1}\sum_{i=j+1}^{k} \frac{\sigma^2}{1-\phi^2}\phi^{i-j} \ . \end{aligned}$$

When $s < k$,

$$\begin{aligned} Cov[I(s), I(k)] &= \sum_{j=1}^{s}\sum_{i=1}^{k} Cov[\delta(j), \delta(i)] \\ &= \sum_{j=1}^{s}\sum_{i=1}^{s} Cov[\delta(j), \delta(i)] + \sum_{j=1}^{s}\sum_{i=s+1}^{k} Cov[\delta(j), \delta(i)] \\ &= Var[I(s)] + \sum_{j=1}^{s}\sum_{i=s+1}^{k} \frac{\sigma^2}{1-\phi^2}\phi^{i-j} \\ &= Var[I(s)] + \frac{\sigma^2}{1-\phi^2}\frac{\phi}{(1-\phi)^2}\left(\phi^s - \phi^k\right)\left(\phi^{-s} - 1\right) . \end{aligned}$$

□

After iteration form (3.1), we can obtain

$$\delta(k) = \delta + \phi^k[\delta(0) - \delta] + \sum_{i=0}^{k} \phi^i \epsilon_{k-i} , \tag{3.6}$$

which can be used to derived the conditional moments of the force of interest given $\delta(0)$. Then we can easily get

$$E[\delta(k) \,|\, \delta(0)] = \delta + \phi^k[\delta(0) - \delta] , \tag{3.7}$$

$$Var[\delta(k) \,|\, \delta(0)] = \frac{\sigma^2}{1-\phi^2}\left(1 - \phi^{2k}\right) , \tag{3.8}$$

$$Cov[\delta(s),\, \delta(k) \,|\, \delta(0)] = \frac{\sigma^2}{1-\phi^2}\phi^{k-s}\left(1 - \phi^{2s}\right) , \tag{3.9}$$

where $s < k$. These expressions are used to obtain the conditional moments of $I(k)$ given $\delta(0) = \delta_0$; the results for the first two moments are presented in the proposition below.

Note that henceforward, we use the simplified notation $E_{\delta_0}[I(k)]$ rather than $E[I(k) \,|\, \delta(0) = \delta_0]$. Similarly, conditional variances and covariances given $\delta(0) = \delta_0$ are expressed as variances and covariances with subscript δ_0.

Proposition 2. *Assume that $\{\delta(j);\, j = 1, 2, \cdots\}$ follows the AR(1) model given by (3.1). Then the 1^{st} and 2^{nd} order conditional moments of $\{I(k);\, k = 1, 2, \cdots\}$ are given by*

$$E_{\delta_0}[I(k)] = E[I(k) \,|\, \delta(0) = \delta_0] = k\delta + (\delta_0 - \delta)\frac{\phi\left(1 - \phi^k\right)}{1-\phi} , \tag{3.10}$$

$$
\begin{aligned}
Var_{\delta_0}[I(k)] &= Var[I(k) \,|\, \delta(0) = \delta_0] \\
&= \frac{\sigma^2}{1-\phi^2}\left[k + \frac{2\phi}{1-\phi}\left(k - 1 - \frac{\phi}{1-\phi}\left(1-\phi^{k-1}\right)\right)\right. \\
&\quad \left. - \left(\frac{\phi}{1-\phi}\right)^2 \left(1-\phi^k\right)^2\right] ,
\end{aligned} \tag{3.11}
$$

and when $s < k,\ s, k = 1, 2, \cdots,$

$$
\begin{aligned}
&Cov_{\delta_0}[I(s), I(k)] = Cov\left[I(s), I(k) \,|\, \delta(0) = \delta_0\right] \\
&= Var_{\delta_0}[I(s)] + \frac{\sigma^2}{1-\phi^2}\frac{\phi(1-\phi^{k-s})}{1-\phi}\left(s - \frac{\phi^2(1-\phi^{2(s-1)})}{1-\phi^2}\right) ,
\end{aligned} \tag{3.12}
$$

where the conditional variance and covariance are independent of the initial value $\delta(0)$.

Proof. Straightforwardly, we have

$$
E[I(k) \,|\, \delta(0)] = E\left[\sum_{j=1}^{k} \delta(j) \,\middle|\, \delta(0)\right] = \sum_{j=1}^{k} E\left[\delta(j) \,|\, \delta(0)\right] ,
$$

$$
\begin{aligned}
Var[I(k) \,|\, \delta(0)] &= Var\left[\sum_{j=1}^{k} \delta(j) \,\middle|\, \delta(0)\right] \\
&= \sum_{j=1}^{k} Var[\delta(j) \,|\, \delta(0)] + 2\sum_{j=1}^{k-1}\sum_{i=j+1}^{k} Cov[\delta(j),\ \delta(j) \,|\, \delta(0)] ,
\end{aligned}
$$

and when $s < k$,

$$
\begin{aligned}
Cov\left[I(s), I(k) \,|\, \delta(0)\right] &= \sum_{j=1}^{s}\sum_{i=1}^{k} Cov[\delta(j), \delta(i) \,|\, \delta(0)] \\
&= \sum_{j=1}^{s}\sum_{i=1}^{s} Cov[\delta(j), \delta(i) \,|\, \delta(0)] + \sum_{j=1}^{s}\sum_{i=s+1}^{k} Cov[\delta(j), \delta(i) \,|\, \delta(0)] \\
&= Var[I(s) \,|\, \delta(0)] + + \sum_{j=1}^{s}\sum_{i=s+1}^{k} Cov[\delta(j), \delta(i) \,|\, \delta(0)] ,
\end{aligned}
$$

where the conditional moments of $\delta(k)$ given $\delta(0)$ are shown in (3.7)- (3.9). Then by letting $\delta(0) = \delta_0$ we get expression (3.10)-(3.12), the 1^{st} and 2^{nd} order conditional moments of $I(k)$ given $\delta(0) = \delta_0$. □

In the next two chapters, we will discuss not only a single insurance policy but also a collection of policies forming an insurance portfolio. Expressions for the expectation and variance of the sum of two $I(k)$'s are needed for the analysis of the life insurance portfolio. The results are given in the proposition below.

Proposition 3. *Assume that* $\{\delta(j);\, j = 1, 2, \cdots\}$ *follows the AR(1) model given by* (3.1). *The conditional mean and variance of the sum of* $I(k_1)$ *and* $I(k_2)$ *are*

$$\begin{aligned} & E_{\delta_0}[I(k_1) + I(k_2)] \\ &= (k_1 + k_2)\delta + (\delta_0 - \delta)\frac{\phi\left(2 - \phi^{k_1} - \phi^{k_2}\right)}{1 - \phi} \ . \end{aligned}$$

and for $k_1 < k_2$,

$$\begin{aligned} & Var_{\delta_0}[I(k_1) + I(k_2)] \\ &= 3Var_{\delta_0}[I(k_1)] + Var_{\delta_0}[I(k_2)] + 2\frac{\sigma^2}{1 - \phi^2}\frac{\phi}{(1 - \phi)^2}\left(\phi^{k_1} - \phi^{k_2}\right)\left(\phi^{-k_1} - 1\right) \ , \end{aligned}$$

where the conditional variance of $I(k)$ *is given in* (3.11)

Proof. With the linearity property of expectation, we have

$$E_{\delta_0}[I(k_1) + I(k_2)] = E_{\delta_0}[I(k_1)] + E_{\delta_0}[I(k_2)] \ .$$

For $k_1 < k_2$, the variance for the sum of two force of interest accumulation functions can be expressed as

$$Var_{\delta_0}[I(k_1) + I(k_2)] = Var_{\delta_0}[I(k_1)] + Var_{\delta_0}[I(k_2)] + 2Cov_{\delta_0}[I(k_1), I(k_2)] \ .$$

Then the results follow from (3.10)- (3.12). □

From (3.6), we have that each force of interest is normally distributed, then the sum is also normally distributed according to the property of the normal distribution, which implies that

$$-I(k) \sim N\Big(-E[I(k)], Var[I(k)]\Big) \qquad (3.13)$$

and

$$-I(k) - I(t) \sim N\Big(- E[I(k)] - E[I(t)], Var[I(k) + I(t)]\Big) \ .$$

Since Y is a random variable with a normal distribution, e^Y is lognormally distributed. Furthermore, the m^{th} moment of e^Y is

$$E\left[e^{mY}\right] = e^{mE[Y]+\frac{m^2}{2}Var[Y]} .$$

This property of the lognormal distribution, together with Eq. (3.13), produces the moments of the discount factor

$$E_{\delta_0}[v(k)] = E_{\delta_0}\left[e^{-I(k)}\right] = e^{-E_{\delta_0}[I(k)]+0.5Var_{\delta_0}[I(k)]} , \quad (3.14)$$

$$E_{\delta_0}[v(k)^2] = E_{\delta_0}\left[e^{-2I(k)}\right] = e^{-2E_{\delta_0}[I(k)]+2Var_{\delta_0}[I(k)]} ,$$

where the conditional expectation and variance of $I(k)$ are given in (3.10) and (3.11), respectively.

Formulas related to the product of two discount factors plays an important role in our portfolio valuation under the stochastic interest rate assumption. The expectation of the product of two discount factors can be determined as below.

$$\begin{aligned} E_{\delta_0}[v(k_1)v(k_2)] &= E_{\delta_0}\left[e^{-I(k_1)-I(k_2)}\right] \\ &= e^{-E_{\delta_0}[I(k_1)-I(k_2)]+0.5\left[Var_{\delta_0}[I(k_1)]+Var_{\delta_0}[I(k_2)]+2Cov_{\delta_0}[I(k_1),I(k_2)]\right]} , \end{aligned}$$

where the conditional covariance of $I(k)$ is given in (3.12). Then expressions related to annuities can be derived based on discount factors as

$$E_{\delta_0}\left[\ddot{a}(k)\right] = E\left[\sum_{j=0}^{k-1} v(j) \,\middle|\, \delta(0) = \delta_0\right] = \sum_{j=0}^{k-1} E_{\delta_0}[v(j)] , \quad (3.15)$$

$$E_{\delta_0}[v(k_1)\ddot{a}(k_2)] = E\left[v(k_1) \sum_{i=0}^{k_2-1} v(i) \,\middle|\, \delta(0) = \delta_0\right] = \sum_{i=0}^{k_2-1} E_{\delta_0}[v(k_1)v(j)] ,$$

$$E_{\delta_0}[\ddot{a}^2(k)] = E\left[\sum_{i=0}^{k-1} v(i) \sum_{j=0}^{k-1} v(j) \,\middle|\, \delta(0) = \delta_0\right] = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} E_{\delta_0}[v(i)v(j)] ,$$

$$E_{\delta_0}[\ddot{a}(k_1)\ddot{a}(k_2)] = E\left[\sum_{i=0}^{k_1-1} v(i) \sum_{j=0}^{k_2-1} v(j) \,\middle|\, \delta(0) = \delta_0\right] = \sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-1} E_{\delta_0}[v(i)v(j)] .$$

3.2 Decrement

We first introduce some basic actuarial notation before describing different mortality assumptions. The symbol (x) is used to denote a life-age-x. Let $T(x)$ be the random future lifetime of (x). To make a probability statement about $T(x)$, we use the following notations when $t \geq 0$,

$$ {}_tq_x = Pr(T(x) \leq t) \ , \tag{3.16} $$

$$ {}_tp_x = 1 - {}_tq_x = Pr(T(x) > t) \ . \tag{3.17} $$

The interpretation of ${}_tq_x$ is the probability that (x) will die within t years, while ${}_tp_x$ is the probability that (x) will survive in the next t years. Let K_x denote the curtate-future-lifetime of (x) which is the number of future years completed by (x) prior to death, K_x is actually the greatest integer in $T(x)$. Then we have the relationship

$$ \begin{aligned} \Pr(K_x = k) &= \Pr(k \leq T(x) < k+1) \\ &= {}_kp_x - {}_{k+1}p_x \qquad (3.18) \\ &= {}_{k+1}q_x - {}_kq_x = {}_{k|}q_x \ . \qquad (3.19) \end{aligned} $$

For the joint life insurance of a pair, we need to calculate the survival and death probabilities involving two individuals. We focuses on two statuses of two lives (x) and (y). The first one is the joint-life status denoted by (xy). It describes a status that exists as long as both in a pair of lives survive and fails upon the first death (see Bowers et al. (1997)). The random future lifetime of the joint-life status (xy) is denoted by $T(xy)$ and $T(xy) = \min(T(x), T(y))$. We have, in actuarial notation,

$$ \begin{aligned} {}_kp_{xy} &= \Pr(T(xy) \geq k) \\ &= \Pr(T(x) \geq k \ , T(y) \geq k) \ . \end{aligned} $$

The other status involving two lives (x) and (y) is the last-survivor status. It describes a status that exists as long as at least one member in a pair of lives is alive and fails upon the last death (see Bowers et al. (1997)). Let $T(\overline{xy})$ be the random future lifetime of the last-survivor status $(\overline{xy})$. It is easy to see that $T(\overline{xy}) = \max(T(x), T(y))$.

$$ \begin{aligned} {}_kq_{\overline{xy}} &= \Pr(T(\overline{xy}) \leq k-1) \\ &= \Pr(0 < T(x) \leq k, 0 < T(y) \leq k) \ . \end{aligned} $$

Formulas presented in (3.16)-(3.18) for a single life can also be extended to the joint-life status and the last-survivor-status cases.

In this project mortality is assumed to be the only cause of decrement for the sake of simplicity. A certain number of mortality models have been presented in the literature review. In this project, the Gompertz distribution is chosen to model the distribution of individual lifetimes. Our choice is simply motivated by its popularity, tractability and the availability of estimates of the parameters of the model. Let X be the age-at-death random variable. We assume that mortality follows a Gompertz distribution, $H(x;\, B,\, c)$, where x is the possible realization of X and B, c are the parameters. Then the Gompertz force of mortality,

$$\mu_x = \frac{H'(x)}{(1 - H(x))} = Bc^x \ ,$$

yields the following distribution of X over $[0, \infty)$:

$$H(x;\, B,\, c) = 1 - e^{\frac{B}{\ln c}(1-c^x)} \ .$$

The Gompertz distribution can be reparameterized to an alternative version which is familiar to statisticians as,

$$H(x;\, m,\, \sigma) = 1 - e^{e^{-m/\sigma}(1-e^{x/\sigma})} \ , \quad x > 0 \ ,$$

with the transformations $B/\mathrm{lnc} = e^{-m/\sigma}$ and $c = e^{1/\sigma}$. The parameter m is the mode and σ is the scale parameter of the distribution. The distribution $H(x;\, m,\, \sigma)$ can be interpreted as the probability that a new born will die before age x, i.e,

$$H(x;\, m,\, \sigma) = Pr(T_0 < x) \ .$$

where T_0 is the random future lifetime of a new born.

The estimates of the parameters in the Gompertz distribution function are taken from Frees et al. (1996), as the detailed estimation procedure is not the focus of this project. A summary of the lifetime distribution for the data used in their paper is provided in Table 3.1. In fact, Frees et al. (1996) analyzed mortality patterns based on information from 14,947 joint and last-survivor annuity contracts in force with a large Canadian insurer over the period December 29, 1988, through December 31, 1993. As pointed out by the authors, three times more males than female died during the study period, implying higher mortality rates for males than females. In addition, their industry data mainly focuses

Table 3.1: Number of Policies by Sex, Entry Age and Mortality Status

Entry Age	Mortality Status: Survive	Mortality Status: Death	Total
Males			
Less than 60	1,170	42	1,212
60-70	7,620	534	8,154
70-80	4,355	806	5,161
Greater than 80	229	177	406
Total	13,374	1,559	14,933
Females			
Less than 60	2,962	30	2,992
60-70	8,222	239	8,461
70-80	3,014	245	3,259
Greater than 80	186	63	249
Total	14,384	577	14,961

on mid-age individuals, therefore the graphs of distribution functions in later sections are conditional on survival to age 40.

Note that the data is left-truncated since deaths are observed only after the contract was in force. This industry data is also censored from the right in the sense that the observation stopped at December 31, 1993. Estimates for univariate and bivariate distributions are obtained by maximum likelihood techniques using the left-truncated, right-censored data. The results of estimation under different mortality assumptions are provided, respectively, at the end of following three sections.

3.2.1 Independent Mortality Assumption

The assumption of independence between lives is commonly used when valuing joint life insurances. The main advantage of this assumption is the simplicity of the calculation. Under this assumption, the joint survival probability can be expressed as the product of individual survival probabilities. Then the problem of the joint survivorship is reduced to a single survivorship problem.

The actuarial notations and probability theories shown in the beginning of Section 3.2 can be applied to the joint life status. For our purpose, let X be the age-at-death for a male newborn, and Y be the age-at-death for a female newborn. We have assumed that X is Gompertz distributed with parameters m_1 and σ_1 while Y is Gompertz distributed with parameters m_2 and σ_2. In the rest of this sections, we use the simplified form $H_1(x)$ rather than $H(x;\, m_1,\, \sigma_1)$ and $H_2(y)$ rather than $H(y;\, m_2,\, \sigma_2)$ to describe male and female individual distributions with different sets of parameters. Then we have the conditional survival probability for (x) and (y),

$$ {}_kp_x = \frac{1 - H(x+k;\, m_1,\, \sigma_1)}{1 - H(x;\, m_1,\, \sigma_1)} = \frac{1 - H_1(x+k)}{1 - H_1(x)} \,, $$

$$ {}_kp_y = \frac{1 - H(y+k;\, m_2,\, \sigma_2)}{1 - H(y;\, m_2,\, \sigma_2)} = \frac{1 - H_2(y+k)}{1 - H_2(y)} \,. $$

With the independent assumption of (x) and (y), the survival probability of (xy) can be obtained in terms of the survival functions $H_1(x)$ and $H_2(y)$ as,

$$ \begin{aligned} {}_kp_{xy} &= {}_kp_x \cdot {}_kp_y \\ &= \frac{1 - H_1(x+k) - H_2(y+k) + H_1(x+k)H_2(y+k)}{1 - H_1(x) - H_2(y) + H_1(x)H_2(y)} \,. \end{aligned} $$

Then the conditional probability that (xy) fails between years $k-1$ and k, ${}_{k-1|}q_{xy}$, can be obtained by using formula (3.18). Similarly, the death probability of $(\overline{xy})$ can be obtained as follows,

$$ \begin{aligned} {}_kq_{\overline{xy}} &= {}_kq_x \cdot {}_kq_y \\ &= \frac{H_1(x+k)H_2(y+k) - H_1(x)H_2(y+k) - H_1(x+k)H_2(y) + H_1(x)H_2(y)}{1 - H_1(x) - H_2(y) + H_1(x)H_2(y)} \,. \end{aligned} $$

Then the conditional probability that $(\overline{xy})$ fails between years $k-1$ and k, ${}_{k-1|}q_{\overline{xy}}$ can be obtained by using (3.19). Estimates of parameters will be given at the end of following two sections for the sake of comparison.

3.2.2 Dependent Mortality Assumption - Copula

According to Nelsen (2006), copulas are referred as functions which join or couple one-dimensional marginal distribution functions to their multivariate distribution functions. Though many possible copula functions have been presented in the literature review, in this project, we use Frank's copula to model the joint life distribution following Frees et al. (1996). Frank's copula is defined as

$$ C(u,v) = \frac{1}{\alpha} \ln\left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\right) \,, \quad 0 \le u \le 1, 0 \le v \le 1 \,, $$

where u and v are two random variable and $\alpha \neq 0$ is a parameter. In the case that u and v are the marginal distribution functions for U and V, the resulting Frank's copula is the joint distribution function for U and V and parameter α measures the dependence between two underlying random variables. A property of copulas proved by Sklar (1959) is that a copula C is uniquely determined if u and v are known and continuous. This property is hold in Frank's copula. Other properties of Frank's copula can be found in Nelsen (1986) and Genest (1987). It can be shown that $\lim_{\alpha \to 0} C(u, v) = uv$, that is, two random variables are independent.

In this case, by using Gompertz distribution as individual's distribution function, the joint distribution can be expressed as

$$\begin{aligned} H(x, y; m_1, \sigma_1, m_2, \sigma_2, \alpha) &= C(H_1(x), H_2(y)) \\ &= \frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha H_1(x)} - 1)(e^{\alpha H_2(y)} - 1)}{e^{\alpha} - 1} \right), \quad x, y > 0\,, \end{aligned}$$

where the marginal distribution for X is Gompertz with parameters m_1 and σ_1, the marginal distribution for Y is also Gompertz with parameters m_2 and σ_2, and α captures the dependence between X and Y. The distribution function $H(x, y; m_1, \sigma_1, m_2, \sigma_2, \alpha)$ can be interpreted as the joint probability that a pair of newborns will die before their ages x and y, i.e.,

$$H(x, y; m_1, \sigma_1, m_2, \sigma_2, \alpha) = Pr(X < x, Y < y)\,.$$

For simplicity, we also use the simplified form $H(x, y)$ for the joint distribution function $H(x, y; m_1, \sigma_1, m_2, \sigma_2, \alpha)$. Then by using the Frank's copula, the survival probability of (xy) can be obtained in terms of the joint distribution function $H(x, y)$ as,

$${}_k p_{xy} = \frac{1 - H(x+k, \infty) - H(\infty, y+k) + H(x+k, y+k)}{1 - H(x, \infty) - H(\infty, y) + H(x, y)}\,.$$

Then the conditional probability that (xy) fails between years $k-1$ and k, ${}_{k-1|}q_{xy}$, can be obtained from (3.18). Similarly, the death probability of $(\overline{xy})$ can also be obtained in terms of the survival function $H(x, y)$ as,

$${}_k q_{\overline{xy}} = \frac{H(x+k, y+k) - H(x, y+k) - H(x+k, y) + H(x, y)}{1 - H(x, \infty) - H(\infty, y) + H(x, y)}\,. \tag{3.20}$$

Then the conditional probability that $(\overline{xy})$ fails between years $k-1$ and k, ${}_{k-1|}q_{\overline{xy}}$ can be obtained from (3.19).

Frees et al. (1996) estimated the model parameters by using the maximum likelihood method with the left-truncated and right-censored data presented in Table 3.1. These estimates are given in Table 3.2 and we will be use them in this project for numerical illustrations.

Table 3.2: Univariate and Bivariate Distributions Parameter Estimates

	Bivariate Distribution		Univariate Distribution	
Parameter	Estimate	Standard Error	Estimate	Standard Error
m_1	85.82	0.26	86.38	0.26
σ_1	9.98	0.40	9.83	0.37
m_2	89.40	0.48	92.17	0.59
σ_2	8.12	0.34	8.11	0.38
α	-3.367	0.346	Not Applicable	Not Applicable

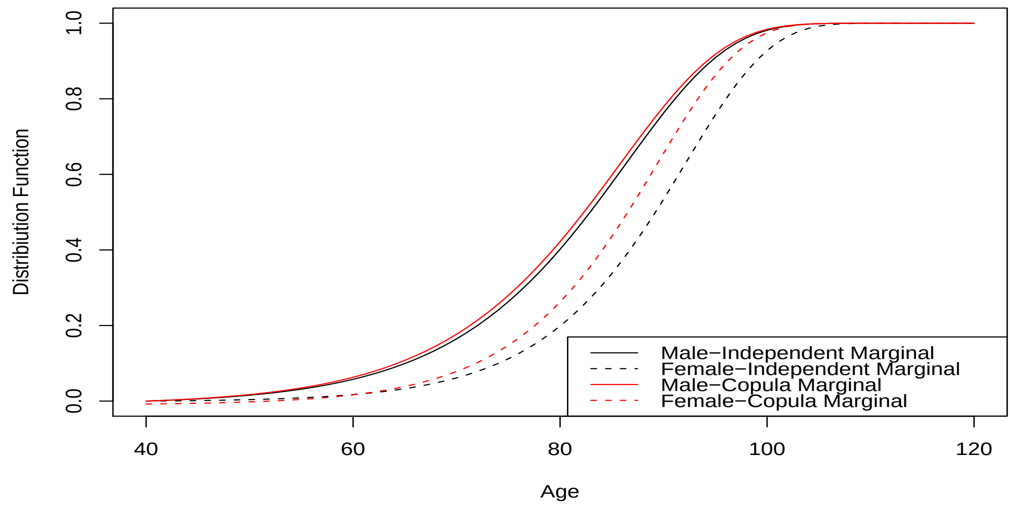


Figure 3.1: Fitted Compertz Distribution Functions Conditional on Survival to Age 40

Figure 3.1 shows the marginal distributions fitted by the Gompertz distribution. It can be observed that the males marginal is above the females marginal, which suggests the higher mortality rates for males than females. Both marginal distributions under the copula model (red line) are above independent marginal distributions (black line) which implies higher mortality rates for individuals under the copula assumption than under the independent assumption.

Figure 3.2 shows the joint distributions under Frank's copula assumption and Figure 3.3 shows the ratio of the copula joint distribution to the independent joint distribution. These joint distributions are conditional on survival to age 40. By looking at the ratio, we can see that the copula joint distribution has higher mortality rates than the joint distribution under the independent assumption.

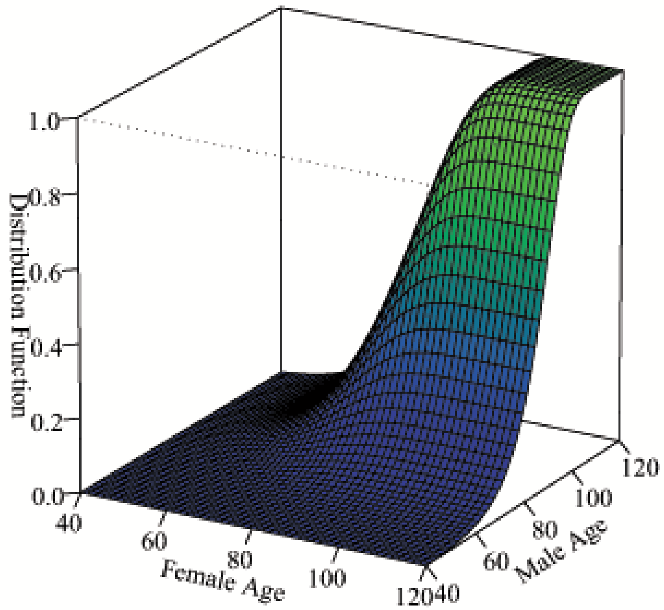


Figure 3.2: Copula Joint Distribution

3.2.3 Dependent Mortality Assumption - Common Shock

While copulas model the future lifetimes of dependent lives through certain functions, a common shock model introduces a common variable (or shock) to each individual's mortality structure. As mentioned in the introduction, this common random variable is used to model some additional hazards from natural catastrophes affecting the individual's mortality risks. The common shock model used in this project is the one presented in Bowers et al. (1997).

Let $T^*(x)$ and $T^*(y)$ be the future lifetime random variables of individuals aged x and y in the absence of the possibility of a common shock, and let Z denote a common shock

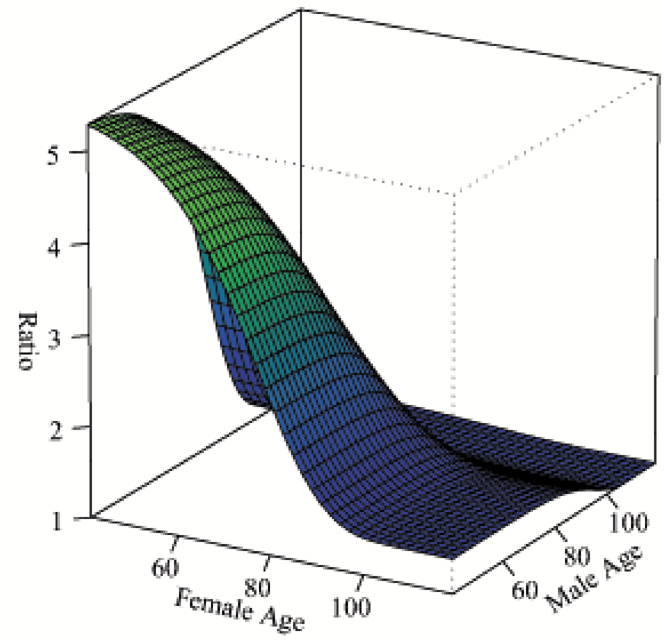


Figure 3.3: Ratio of Copula Joint Distribution to Independent Joint Distribution

random variable. Let $T(x)$ and $T(y)$ be the corresponding underlying future lifetime random variables incorporating the common shock variable Z. Define $T(x) = \min[T^*(x), Z]$ and $T(y) = \min[T^*(y), Z]$. Two assumptions are made in the common shock model; Z is independent of $T^*(x)$ and $T^*(y)$ and is exponentially distributed with parameter λ, that is, $\Pr(Z > t) = e^{-\lambda t}$. Under these assumptions, we can obtain the survival function of the joint-life of $T(x)$ and $T(y)$ as

$$\begin{aligned} {}_tp_{xy} &= \Pr\left(T(x) > t, T(y) > t\right) \\ &= \Pr\left(\min[T^*(x), Z] > t, \min[T^*(y), Z] > t\right) \\ &= \Pr\left(T^*(x) > t, T^*(y) > t, Z > t\right) \\ &= {}_tp_x^*\, {}_tp_y^*\, e^{-\lambda t} \\ &= \frac{(1 - H_1(x+t))(1 - H_2(y+t))}{(1 - H_1(x))(1 - H_2(y))}\, e^{-\lambda t}\,. \end{aligned} \tag{3.21}$$

The survival probability for the last-survivor status can be obtained from

$$\begin{aligned} {}_kp_{\overline{xy}} &= {}_kp_x + {}_kp_y - {}_kp_{xy} \\ &= {}_kp_x + {}_kp_y - {}_kp_x^* {}_kp_y^* \, e^{-\lambda k} \, . \end{aligned} \tag{3.22}$$

Then the conditional probability that (xy) fails between years $k-1$ and k, ${}_{k-1|}q_{xy}$, can be obtained from (3.18), while the conditional probability that $(\overline{xy})$ fails between years $k-1$ and k, ${}_{k-1|}q_{\overline{xy}}$, can be obtained by using the formula (3.19).

The same data set presented in Table 3.1 was used to get the estimates of the parameters for the common shock mortality model. The results of the common shock estimations are given in Table 3.3 (see Frees et al. (1996)).

Table 3.3: Common Shock Parameter Estimates

	Bivariate Distribution		Univariate Distribution	
Parameter	Estimate	Standard Error	Estimate	Standard Error
m_1	86.66	0.27	86.38	0.26
σ_1	9.98	0.37	9.83	0.37
m_2	92.69	0.64	92.17	0.59
σ_2	8.09	0.40	8.11	0.38
λ	0.00054	0.00010	Not Applicable	Not Applicable

3.3 Other Assumptions

In the following chapters, we will analyze a portfolio of joint life insurance with two lives involved, which pays the benefit upon the first death or the second death. The types of insurance considered are discrete term life insurance and endowment insurance. Consider a portfolio with m joint first-to-die insurance policies. The benefit is paid at the end of the year in which the first death occurs, called the joint first-to-die insurance benefit. Let $\{K_{x_iy_i}\}$ denote the curtate-future-lifetime of the i^{th} joint first-to-die policy in the portfolio issued at age of (x_i, y_i). We make the following assumptions for random variables $\{K_{x_iy_i};\, i = 1, 2, \cdots\}$ for this project:

1. Random variables $\{K_{x_iy_i};\, i = 1, 2, \cdots\}$ are mutually independent.

2. Random variables $\{K_{x_iy_i};\, i = 1, 2, \cdots\}$ and $\{\delta(k),\, k = 0, 1, \cdots\}$ are mutually independent.

3. The conditional loss random variables given $\{\delta(k),\, k = 0, 1, \cdots\}$ are independent.

The corresponding assumptions can be made for a portfolio with a number of joint last-to-die insurance policies.

The first assumption assures that the future lifetime of each pair in the portfolio are independent. The independency between the mortality and the interest process is guaranteed by the second assumption. Policies in the same portfolio are not mutually independent because of the same discounting factor. The third assumption means that individual policies in the portfolio can be independent after conditioning on the interest rate process.

Chapter 4

Joint First-To-Die Insurance Valuation

Joint life or multi-life insurance products insure two or more lives in one policy. In this projet, we limit our discussion to insurance which cover two lives in the same policy. Joint life products can be in as many forms as single life products, for instance, term life insurance, endowment life insurance, annuity, etc. There are two types of life insurance covering a pair of lives:

1. Joint first-to-die products which pay the death benefit when the first death occurs, called joint-life insurance.

2. Joint last-to-die or joint last-to-die products pay the death benefit when the second death occurs, called last-survivor insurance.

We focus on the first type of life insurance in this chapter, and discuss the second case in the next chapter. Moreover in this project, we study a joint first-to-die insurance on (xy) in which

- the death benefit, b, is payable at the end of the year of the first death before time n,

- the pure endowment benefit c is payable at the end of the year if both survive to year n,

- level premium, π, is paid annually, at the beginning of each policy year if both survive until year n,

- the level premium is determined using the equivalence principle.

We call this special insurance an n-year joint first-to-die endowment insurance.

4.1 Single policy valuation

In this section, the investigation of loss variables is done by the prospective method. Suppose that we value this joint first-to-die insurance at time r, that is, at the beginning of the $(r+1)^{th}$ year. The prospective loss at time r is defined as the net difference between the present value at time r of future benefits and the present value at time r of future premiums. Let K_{xy} denote the curtate-future-lifetime of the joint-life status (xy). For a non-negative integer r, let ${}_rL_{xy}$ denote the conditional prospective loss random variable at time r for an n-year endowment joint first-to-die insurance contract issued at the age of (x,y), given that the joint-life status (xy) survives to time r. The loss random variable can be expressed as a function of $K_{x+r:y+r}$, it is

$$ {}_rL_{xy} = \begin{cases} bv(K_{x+r:y+r}+1) - \pi\ddot{a}(K_{x+r:y+r}+1), & K_{x+r:y+r} = 0, 1, \cdots, n-r-1, \\ cv(n-r) - \pi\ddot{a}(n-r), & K_{x+r:y+r} = n-r, n-r+1, \cdots. \end{cases} \tag{4.1} $$

By including zero values for K_{xy} less than r, we can extend the definition given in (4.1) to the unconditional one, ${}_rL^{uc}_{xy}$, as

$$ {}_rL^{uc}_{xy} = \begin{cases} 0, & K_{xy} = 1, 2\cdots, r-1, \\ bv(K_{xy}-r+1) - \pi\ddot{a}(K_{xy}-r+1), & K_{xy} = r, r+1, \cdots, n-1, \\ cv(n-r) - \pi\ddot{a}(n-r), & K_{xy} = n, n+1, \cdots. \end{cases} $$

With the assumption that the interest rates follow an AR(1) process and the current rate at time r is δ_0, the β^{th} moment of the conditional prospective random variable ${}_rL_{xy}$, given that the joint-life status (xy) survives to time r, can be calculated directly from the definition of ${}_rL_{xy}$ given in (4.1) by conditioning on the survivorship of (xy) as follows:

$$\begin{aligned} E_{\delta_0}\left[({}_rL_{xy})^\beta\right] &= E_{\delta_0}\left[({}_rL_{xy}^{uc})^\beta \,|\, K_{xy} \geq r\right] \\ &= E\left[E\left[({}_rL_{xy})^\beta \,|\, K_{xy},\, \delta(r) = \delta_0\right]\right] \\ &= \sum_{k=0}^{n-r-1} E_{\delta_0}\left[\left(bv(k+1) - \pi\ddot{a}(k+1)\right)^\beta\right] \cdot {}_{k|}q_{x+r:y+r} \\ &\quad + E_{\delta_0}\left[\left(cv(n-r) - \pi\ddot{a}(n-r)\right)^\beta\right] \cdot {}_{n-r}p_{x+r:y+r} \,. \end{aligned} \tag{4.2}$$

With $\beta = 1$, we have the expectation of the conditional prospective loss random variable ${}_rL_{xy}$,

$$\begin{aligned} E_{\delta_0}[{}_rL_{xy}] = &\sum_{k=0}^{n-r-1} \Big(b\, E_{\delta_0}[v(k+1)] - \pi\, E_{\delta_0}[\ddot{a}(k+1)]\Big)_{k|}q_{x+r:y+r} \\ &+ \Big(c\, E_{\delta_0}[v(n-r)] - \pi\, E_{\delta_0}[\ddot{a}(n-r)]\Big)_{n-r}p_{x+r:y+r} \,. \end{aligned} \tag{4.3}$$

Similarly, the conditional second moment can be obtained by setting $\beta = 2$ in (4.2) as

$$\begin{aligned} E_{\delta_0}\left[({}_rL_{xy})^2\right] = &\sum_{k=0}^{n-r-1} \Big(b^2\, E_{\delta_r}[v(k+1)^2] - 2b\pi\, E_{\delta_0}[v(k+1)\ddot{a}(k+1)] \\ &+ \pi^2 E_{\delta_0}[\ddot{a}(k+1)^2]\Big) \cdot {}_{k|}q_{x+r:y+r} + \Big(c^2\, E_{\delta_0}[v(n-r)^2] \\ &- 2c\pi\, E_{\delta_0}[v(n-r)\ddot{a}(n-r)] + \pi^2 E_{\delta_0}[\ddot{a}(n-r)^2]\Big) \cdot {}_{n-r}p_{x+r:y+r} \,. \end{aligned} \tag{4.4}$$

Then the variance of ${}_rL_{xy}$ can be obtained by:

$$Var_{\delta_0}[{}_rL_{xy}] = E_{\delta_0}[({}_rL_{xy})^2] - E_{\delta_0}[{}_rL_{xy}]^2 \,.$$

Furthermore, we can calculate the β^{th} moment of unconditional prospective random variable ${}_rL_{xy}^{uc}$ by the law of total probability. Given that the current rate at time r is δ_0, we have

$$\begin{aligned} &E_{\delta_0}\left[({}_rL_{xy}^{uc})^\beta\right] \\ &= E_{\delta_0}\left[({}_rL_{xy}^{uc})^\beta \,|\, K_{xy} \leq r-1\right] \cdot Pr[K_{xy} \leq r-1] + E_{\delta_0}\left[({}_rL_{xy}^{uc})^\beta \,|\, K_{xy} \geq r\right] \cdot Pr[K_{xy} \geq r] \\ &= 0 \cdot {}_rq_{xy} + E_{\delta_0}\left[({}_rL_{xy}^{uc})^\beta \,|\, K_{xy} \geq r\right] \cdot {}_rp_{xy} \\ &= E_{\delta_0}\left[({}_rL_{xy})^\beta\right] \cdot {}_rp_{xy} \,. \end{aligned}$$

Note that in the rest of this project, we focus on the conditional prospective loss random variable, simply called the loss random variable.

4.2 First Expression for the Prospective Loss Variables for Non-homogeneous Portfolio

Now we investigate the property of the loss random variable for a general (non-homogeneous) portfolio. At a given valuation date, consider a non-homogeneous portfolio with m joint first-to-die endowment insurance contracts, in the sense that these contracts might have different terms n_i, different death benefits b_i and different pure endowment benefits c_i for $i = 1, 2, \cdots, m$. The i^{th} joint first-to-die endowment insurance contract is issued to a pair aged (x_i, y_i). The specific valuation time for contract i is r_i, that is the beginning of the $(r_i+1)^{th}$ policy year. Now let ${}_{r_i}L_{x_iy_i}$ be the loss random variable at time r_i for the i^{th} n_i-year endowment life insurance contract issued to a pair aged (x_i, y_i) with premium π_i, death benefit b_i, and pure endowment benefit c_i at time n_i, $i = 1, 2, \cdots, m$. Note that ${}_{r_i}L_{x_iy_i}$ is the conditional prospective loss random variable given that the joint-life status (x_i, y_i) has survived r_i years and the survival times vary from policy to policy. Let L^P denote the loss random variable for the whole portfolio, namely,

$$L^P = \sum_{i=1}^{m} {}_{r_i}L_{x_iy_i} \ ,$$

where ${}_{r_i}L_{x_iy_i}$ is in the form of

$${}_{r_i}L_{x_iy_i} = \begin{cases} b_i v(K_{x_i+r_i:y_i+r_i}+1) - \pi_i \ddot{a}(K_{x_i+r_i:y_i+r_i}+1), & K_{x_i+r_i:y_i+r_i} = 0, 1, \cdots, n_i - r_i - 1, \\ c_i v(n_i - r_i) - \pi_i \ddot{a}(n_i - r_i), & K_{x_i+r_i:y_i+r_i} = n_i - r_i, \cdots, \end{cases}$$

Here the individual prospective loss random variables, ${}_{r_i}L_{x_iy_i}$'s, are dependent because of the same interest rate process.

We make the assumption that the interest rates follow a conditional AR(1) process with the initial interest rate at our specific valuation time being δ_0. By (4.3), the expression for $E_{\delta_0}[{}_{r_i}L_{x_iy_i}]$ is given by

$$\begin{aligned} E_{\delta_0}\left[{}_{r_i}L_{x_iy_i}\right] = & \sum_{k=0}^{n_i-r_i-1} \Big(b_i\, E_{\delta_0}[v(k+1)] - \pi_i\, E_{\delta_0}[\ddot{a}(k+1)]\Big)\, {}_{k|}q_{x_i+r_i:y_i+r_i} \\ & + \Big(c_i\, E_{\delta_0}[v(n_i-r_i)] - \pi_i\, E_{\delta_0}[\ddot{a}(n_i-r_i)]\Big)\, {}_{n_i-r_i}p_{x_i+r_i:y_i+r_i} \ . \end{aligned} \tag{4.5}$$

In order to derive expressions for the variance, we first note that for the expectation of the product of two loss random variables, ${}_{r_i}L_{x_iy_i}$ and ${}_{r_j}L_{x_jy_j}$, there are four possibilities; both

$(x_i + r_i, y_i + r_i)$ and $(x_j + r_j, y_j + r_j)$ fail before n_i and n_j, both survive to the end of the terms of the contracts, $(x_i + r_i, y_i + r_i)$ fails before n_i and $(x_j + r_j, y_j + r_j)$ survives to n_j or vice verse. Combining all the terms under these four possibilities we have

$$\begin{aligned}
&E_{\delta_0}\left[{}_{r_i}L_{x_iy_i}\cdot{}_{r_j}L_{x_jy_j}\right]\\
&=\sum_{k_i=0}^{n_i-r_i-1}\sum_{k_j=0}^{n_j-r_j-1}\left\{\begin{array}{c} b_ib_jE_{\delta_0}[v(k_i+1)v(k_j+1)]\\ +\pi_i\pi_jE_{\delta_0}[\ddot{a}(k_i+1)\ddot{a}(k_j+1)]\\ -b_i\pi_jE_{\delta_0}[v(k_i+1)\ddot{a}(k_j+1)]\\ -\pi_ib_jE_{\delta_0}[\ddot{a}(k_i+1)v(k_j+1)]\end{array}\right\}{}_{k_i|}q_{x_i+r_i:y_i+r_i}\cdot{}_{k_j|}q_{x_j+r_j:y_j+r_j}\\
&+\sum_{k_i=0}^{n_i-r_i-1}\left\{\begin{array}{c} b_ic_jE_{\delta_0}[v(k_i+1)v(n_j-r_j)]\\ +\pi_i\pi_jE_{\delta_0}[\ddot{a}(k_i+1)\ddot{a}(n_j-r_j)]\\ -b_i\pi_jE_{\delta_0}[v(k_i+1)\ddot{a}(n_j-r_j)]\\ -\pi_ic_jE_{\delta_0}[\ddot{a}(k_i+1)v(n_j-r_j)]\end{array}\right\}{}_{k_i|}q_{x_i+r_i:y_i+r_i}\cdot{}_{n_j-r_j}p_{x_j+r_j:y_j+r_j}\\
&+\sum_{k_j=0}^{n_j-r_j-1}\left\{\begin{array}{c} b_jc_iE_{\delta_0}[v(k_j+1)v(n_i-r_i)]\\ +\pi_j\pi_iE_{\delta_0}[\ddot{a}(k_j+1)\ddot{a}(n_i-r_i)]\\ -b_j\pi_iE_{\delta_0}[v(k_j+1)\ddot{a}(n_i-r_i)]\\ -\pi_jc_iE_{\delta_0}[\ddot{a}(k_j+1)v(n_i-r_i)]\end{array}\right\}{}_{k_j|}q_{x_j+r_j:y_j+r_j}\cdot{}_{n_i-r_i}p_{x_i+r_i:y_i+r_i}\\
&+\left\{\begin{array}{c} c_ic_jE_{\delta_0}[v(n_i-r_i)v(n_j-r_j)]\\ +\pi_i\pi_jE_{\delta_0}[\ddot{a}(n_i-r_i)\ddot{a}(n_j-r_j)]\\ -c_i\pi_jE_{\delta_0}[v(n_i-r_i)\ddot{a}(n_j-r_j)]\\ -\pi_ic_jE_{\delta_0}[\ddot{a}(n_i-r_i)v(n_j-r_j)]\end{array}\right\}{}_{n_i-r_i}p_{x_i+r_i:y_i+r_i}\cdot{}_{n_j-r_j}p_{x_j+r_j:y_j+r_j}\ . \qquad (4.6)
\end{aligned}$$

Similarly, the second moment can be obtained from (4.4) by replacing the set of parameters $\{x, y, r, b, c, n\}$ by $\{x_i, y_i, r_i, b_i, c_i, n_i\}$, that is,

$$\begin{aligned}
&E_{\delta_0}[{}_{r_i}L^2_{x_iy_i}]\\
&=\sum_{k=0}^{n_i-r_i-1}\Big(b_i^2\,E_{\delta_0}[v(k+1)^2]-2b_i\pi_i\,E_{\delta_0}[v(k+1)\ddot{a}(k+1)]+\pi_i^2E_{\delta_0}[\ddot{a}(k+1)^2]\Big){}_{k|}q_{x_i+r_i:y_i+r_i}\\
&+\Big(c_i^2\,E_{\delta_0}[v(n-r)^2]-2c_i\pi_i\,E_{\delta_0}[v(n_i-r_i)\ddot{a}(n_i-r_i)]\\
&+\pi_i^2E_{\delta_0}[\ddot{a}(n_i-r_i)^2]\Big){}_{n_i-r_i}p_{x_i+r_i:y_i+r_i}\ . \qquad (4.7)
\end{aligned}$$

Using (4.5)- (4.7), expressions for $Var_{\delta_0}[{}_{r_i}L_{x_iy_i}]$ and $Cov_{\delta_0}\left[{}_{r_i}L_{x_iy_i}, {}_{r_j}L_{x_jy_j}\right]$ can be derived from

$$Var_{\delta_0}[{}_{r_i}L_{x_iy_i}] = E_{\delta_0}[{}_{r_i}L_{x_iy_i}]^2 - E_{\delta_0}[{}_{r_i}L^2_{x_iy_i}] \ , \tag{4.8}$$

$$Cov_{\delta_0}[{}_{r_i}L_{x_iy_i} \ , {}_{r_j}L_{x_jy_j}] = E_{\delta_0}[{}_{r_i}L_{x_iy_i} \, {}_{r_j}L_{x_jy_j}] - E_{\delta_0}[{}_{r_i}L_{x_iy_i}]E_{\delta_0}[{}_{r_j}L_{x_jy_j}] \ . \tag{4.9}$$

Therefore, the expression for $E_{\delta_0}[L^P]$ is derived by summing the expectation of individual loss random variable, $E_{\delta_0}[{}_{r_i}L_{x_iy_i}]$, given in (4.5) for all policies, as

$$E_{\delta_0}\left[L^P\right] = E_{\delta_0}\left[\sum_{i=1}^{m} {}_{r_i}L_{x_iy_i}\right] = \sum_{i=1}^{m} E_{\delta_0}\left[{}_{r_i}L_{x_iy_i}\right] \ .$$

The expression for $Var_{\delta_0}[L^P]$ is determined by

$$\begin{aligned} Var_{\delta_0}\left[L^P\right] &= Var_{\delta_0}\left[\sum_{i=1}^{m} {}_{r_i}L_{x_iy_i}\right] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} Cov_{\delta_0}[{}_{r_i}L_{x_iy_i}, \ {}_{r_j}L_{x_jy_j}] \\ &= \sum_{i=1}^{m} Var_{\delta_0}[{}_{r_i}L_{x_iy_i}] + \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} Cov_{\delta_0}[{}_{r_i}L_{x_iy_i}, \ {}_{r_j}L_{x_jy_j}] \end{aligned}$$

where the corresponding variance and covariance expressions are given by (4.8) and (4.9).

4.3 Second Expression for the Prospective Loss Variables for Non-homogeneous Portfolio

In the previous section, we studied the loss random variable for the whole portfolio by studying the loss random variable for each single contract. Then the moments of the loss random variable for the whole portfolio were obtained by combining the moments of the loss random variable for each single contract in the portfolio. In this section we use an alternative way to study the loss random variable for a portfolio used by Parker (1997) and Marceau and Gaillardetz (1999). This method focuses on the annual payouts of the portfolio which is useful to approximate the distribution of the loss random variable for the portfolio.

For the same non-homogeneous portfolio studied in the preview section, let $CF(j)$ be the random cash flow payable at time j with respect to the whole portfolio. The terms and the number of years survived at our valuation time (n_i and r_i for policy i) vary from policy to policy in the portfolio and the latest possible cash flow for the portfolio may occur at time n, where $n = \max_{1 \le i \le m}(n_i - r_i)$. Actually the cash flow $CF(j)$ is the net difference between benefits and the premiums paid at time j, $j = 1, 2, \cdots, n$, that is,

$$CF(j) = \sum_{i=1}^{m} D_{i,j}\, b_i\, 1_{(n_i - r_i \ge j)} - \sum_{i=1}^{m} S_{i,j}\, \pi_i\, 1_{(n_i - r_i > j)} + \sum_{i=1}^{m} S_{i,n_i - r_i}\, c_i\, 1_{(n_i - r_i = j)} \ , \tag{4.10}$$

where the random variables $D_{i,j}$ and $S_{i,j}$ are defined by

$$D_{i,j} = \begin{cases} 1, & \text{if the first death in policy i occurs within the interval } (j-1, j]\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

$$S_{i,j} = \begin{cases} 1, & \text{if both insureds in policy i survive to time j}\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

For j=0, the expression is

$$CF(0) = -\sum_{i=1}^{m} S_{i,0}\, \pi_i \ ,$$

Note that $D_{i,0} = 0$ for any i. Then it is easy to see, for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, that

$$E[D_{i,j}] = {}_{j-1|}q_{x_i + r_i : y_i + r_i} \ , \tag{4.11}$$

$$E[S_{i,j}] = {}_{j}p_{x_i + r_i : y_i + r_i} \ , \tag{4.12}$$

$$Var[D_{i,j}] = {}_{j-1|}q_{x_i + r_i : y_i + r_i} \left(1 - {}_{j-1|}q_{x_i + r_i : y_i + r_i}\right) \ , \tag{4.13}$$

$$Var[S_{i,j}] = {}_{j}p_{x_i + r_i : y_i + r_i} \left(1 - {}_{j}p_{x_i + r_i : y_i + r_i}\right) \ . \tag{4.14}$$

For $0 \le k < j$ and $i = 1, 2, \cdots, m$, we have the following covariances

$$Cov[D_{i,k}, D_{i,j}] = -{}_{k-1|}q_{x_i + r_i : y_i + r_i} \cdot {}_{j-1|}q_{x_i + r_i : y_i + r_i} \ , \tag{4.15}$$

$$Cov[D_{i,j}, S_{i,j}] = -{}_{j-1|}q_{x_i + r_i : y_i + r_i} \cdot {}_{j}p_{x_i + r_i : y_i + r_i} \ , \tag{4.16}$$

$$Cov[D_{i,k}, S_{i,j}] = -{}_{k-1|}q_{x_i + r_i : y_i + r_i} \cdot {}_{j}p_{x_i + r_i : y_i + r_i} \ , \tag{4.17}$$

$$Cov[D_{i,j}, S_{i,k}] = {}_{j-1|}q_{x_i + r_i : y_i + r_i} \cdot \left(1 - {}_{k}p_{x_i + r_i : y_i + r_i}\right) , \tag{4.18}$$

$$Cov[S_{i,j}, S_{i,k}] = {}_{j}p_{x_i + r_i : y_i + r_i} \cdot \left(1 - {}_{k}p_{x_i + r_i : y_i + r_i}\right) . \tag{4.19}$$

Based on the assumptions in Section 3.3, the random variables $\{K_{x_i y_i};\, i = 1, 2, \cdots\}$ are mutually independent. Therefore, all the covariances above are zero for a pair of variables from two different policies. For instance, considering two policies i_1 and i_2 with $i_1 \ne i_2$, we have $Cov[D_{i_1,k}, D_{i_2,j}] = 0$ for any k and j.

The relationship between the first expression for the prospective loss variables studied in Section 4.2 and the one studied in this section for a non-homogeneous portfolio can be described as

$$L^P = \sum_{j=0}^{n} CF(j)v(j) = \sum_{i=1}^{m} {}_{r_i}L_{x_i y_i}\ , \tag{4.20}$$

where the former sums over all the possible policy years and the latter sums over all the policies.

By taking the expectation of both sides of (4.10) and using (4.11)-(4.12), we obtain the expectation of the cash flow paid at time j, $j = 1, 2, \cdots, n$,

$$\begin{aligned} E[CF(j)] = & \sum_{i=1}^{m} {}_{j-1|}q_{x_i+r_i:y_i+r_i}\, b_i\, 1_{(n_i-r_i \geq j)} - \sum_{i=1}^{m} {}_{j}p_{x_i+r_i:y_i+r_i}\, \pi_i\, 1_{(n_i-r_i > j)} \\ & + \sum_{i=1}^{m} {}_{n_i-r_i}p_{x_i+r_i:y_i+r_i}\, c_i\, 1_{(n_i-r_i = j)}\ , \end{aligned} \tag{4.21}$$

and for j=0, the expression is

$$E[CF(0)] = CF(0) = -\sum_{i=1}^{m} \pi_i.$$

The variance and covariance of the cash flow at time j, $j = 1, 2, \cdots, n$n, can be obtained using (4.11)- (4.19) as follows:

$$\begin{aligned} Var[CF(j)] = & \sum_{i=1}^{m} b_i^2\, Var[D_{i,j}]\, 1_{(n_i-r_i \geq j)} + \sum_{i=1}^{m} c_i^2\, Var[S_{i,n_i-r_i}]\, 1_{(n_i-r_i = j)} \\ & +2\sum_{i=1}^{m} b_i c_i\, Cov[D_{i,n_i-r_i}, S_{i,n_i-r_i}]\, 1_{(n_i-r_i=j)} + \sum_{i=1}^{m} \pi_i^2\, Var[S_{i,j}]\, 1_{(n_i-r_i>j)} \\ & -2\sum_{i=1}^{m} \pi_i b_i\, Cov[D_{i,j}, S_{i,j}] 1_{(n_i-r_i>j)}\ , \end{aligned} \tag{4.22}$$

and for $k < j$ with $j = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, n$,

$$\begin{aligned} & Cov[CF(k), CF(j)] \\ = & \sum_{i=1}^{m} b_i^2\, Cov[D_{i,k}, D_{i,j}]\, 1_{(n_i-r_i \geq j)} + \sum_{i=1}^{m} b_i c_i\, Cov[D_{i,k}, S_{i,n_i-r_i}]\, 1_{(n_i-r_i=j)} \\ & - \sum_{i=1}^{m} b_i \pi_i\, Cov[D_{i,k}, S_{i,j}]\, 1_{(n_i-r_i>j)} + \sum_{i=1}^{m} \pi_i \pi_j\, Cov[S_{i,k}, S_{i,j}]\, 1_{(n_i-r_i>j)} \\ & - \sum_{i=1}^{m} b_i \pi_i\, Cov[S_{i,k}, D_{i,j}]\, 1_{(n_i-r_i \geq j)} - \sum_{i=1}^{m} \pi_i c_i\, Cov[S_{i,k}, S_{i,n_i-r_i}]\, 1_{(n_i-r_i=j)}\ . \end{aligned} \tag{4.23}$$

By summing the products of the expectations of the cash flows and the discount factors under assumption 2 made in Section 3.3, that is, "random variables $\{K_{x_iy_i};\ i = 1, 2, \cdots\}$ and $\{\delta(k);\ k = 0, 1, \cdots\}$ are mutually independent", we have the following expression for $E_{\delta_0}\left[L^P\right]$:

$$E_{\delta_0}\left[L^P\right] = E\left[\sum_{j=0}^{n} CF(j)v(j) \,\Big|\, \delta(0) = \delta_0\right] = \sum_{j=0}^{n} E[CF(j)]E_{\delta_0}[v(j)] \ . \tag{4.24}$$

According to Parker (1997), the variance of the loss random variable of the whole portfolio, $Var_{\delta_0}\left[L^P\right]$, can be split into two components, corresponding to the insurance risk and the investment risk. Since $Var_{\delta_0}\left[L^P\right]$ is a function of two random process; remaining future lifetimes $\{K_{x_iy_i}; i = 1, \cdots, m\}$ and the force of interest $\{\delta(k); k = 1, \cdots, n\}$, there are two ways of getting the variance. In the following, we use $\{K_{x_iy_i}\}$ and $\{\delta(k)\}$ as the simplified form of $\{K_{x_iy_i}; i = 1, \cdots, m\}$ and $\{\delta(k), k = 1, \cdots, n\}$. First by conditioning on $\{K_{x_iy_i}\}$, the variance is shown as:

$$Var_{\delta_0}\Big[L^P\Big] = E\Big[Var\Big[L^P \,\Big|\, \{K_{x_iy_i}\},\, \delta(0) = \delta_0\Big]\Big] + Var\Big[E\Big[L^P \,\Big|\, \{K_{x_iy_i}\},\, \delta(0) = \delta_0\Big]\Big] \ , \tag{4.25}$$

where

$$\begin{aligned} E\Big[Var\Big[L^P \,\Big|\, \{K_{x_iy_i}\},\, \delta(0) = \delta_0\Big]\Big] &= E\left[Var\left[\sum_{j=1}^{n} CF(j)\, v(j) \,\Bigg|\, \{K_{x_iy_i}\},\, \delta(0) = \delta_0\right]\right] \\ &= E\left[\sum_{j=1}^{n}\sum_{k=1}^{n} CF(j)\, CF(k)\, Cov_{\delta_0}[v(j), v(k)]\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} E[CF(j)\, CF(k)]\ Cov_{\delta_0}[v(j), v(k)] \ , \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} Var\Big[E\Big[L^P \,\Big|\, \{K_{x_iy_i}\},\, \delta(0) = \delta_0\Big]\Big] &= Var\left[E\left[\sum_{j=1}^{n} CF(j)\ v(j) \,\Bigg|\, \{K_{x_iy_i}\},\, \delta(0) = \delta_0\right]\right] \\ &= Var\left[\sum_{j=1}^{n} CF(j)\ E_{\delta_0}[v(j)]\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} E_{\delta_0}[v(j)]\ E_{\delta_0}[v(k)]\ Cov[CF(j), CF(k)]. \end{aligned} \tag{4.27}$$

Alternatively, we can also write the variance by conditioning on $\{\delta(k)\}$ as

$$Var_{\delta_0}\Big[L^P\Big] = E\Big[Var\Big[L^P \;\Big|\; \{\delta(k)\}\Big]\Big] + Var\Big[E\Big[L^P \;\Big|\; \{\delta(k)\}\Big]\Big] \,, \tag{4.28}$$

where

$$\begin{aligned} E\Big[Var\Big[L^P \;\Big|\; \{\delta(k)\}\Big]\Big] &= E\left[Var\left[\sum_{j=1}^{n} CF(j)\, v(j) \;\middle|\; \{\delta(k)\}\right]\right] \\ &= E\left[\sum_{j=1}^{n}\sum_{k=1}^{n} v(j)\, v(k)\, Cov[CF(j), CF(k)]\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} E_{\delta_0}[v(j)\, v(k)]\; Cov[CF(j), CF(k)] \,, \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} Var\Big[E\Big[L^P \;\Big|\; \{\delta(k)\}\Big]\Big] &= Var\left[E\left[\sum_{j=1}^{n} CF(j)\; v(j) \;\middle|\; \{\delta(k)\}\right]\right] \\ &= Var\left[\sum_{j=1}^{n} v(j)\; E[CF(j)]\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} E[CF(j)]\; E[CF(k)]\; Cov_{\delta_0}[v(j), v(k)] \,. \end{aligned} \tag{4.30}$$

In (4.25), we decompose $Var_{\delta_0}\left[L^P\right]$ into (4.26) and (4.27) by conditioning on $\{K_{x_i y_i}\}$, that is, fixing the cash flows of the portfolio. The expectation of conditional variance given in (4.26) is the average over cash flows where the riskiness of cash flows has been averaged out. Therefore (4.26) is a measure of the variability of L^P caused by stochastic interest rates, that is, the investment risk. With the same logic, (4.27) which is the variance of conditional expectation measures the insurance risk. For the alternative expression of $Var_{\delta_0}\left[L^P\right]$ given by (4.28), we can also make a similar argument that (4.29) represents the insurance risk while (4.30) represents the investment risk.

Chapter 5

Joint Last-To-Die Life Insurance Valuation

Joint last-survivor or joint last-to-die products pay the death benefit when the second death occurs. After the discussion of joint first-to-die insurance in Chapter 4, we focus on the valuation of joint last-to-die life insurance which covers a pair of lives in this chapter.

Suppose that a joint last-to-die insurance is issued to a pair aged (x, y) which pays a death benefit b at the end of the year of the second death before time n or a pure endowment benefit c if any in the pair survives to year n, called the n-year joint last-to-die endowment life insurance. A level premium, π, is paid at the beginning of each year from time r to $n-1$, or until the failure of the last-survivor status $(\overline{xy})$ if earlier than n. The level premium is determined using the equivalence principle.

5.1 Single policy valuation

In this section, the investigation of the loss variable is also done by the prospective method. Suppose that we value this joint last-to-die insurance at time r, that is, at the beginning of $(r+1)^{th}$ year. Let ${}_rL_{\overline{xy}}$ be the conditional prospective loss random variable at time r for an n-year joint endowment last-to-die life insurance contract issued at their ages of (x, y) given that the last-survivor status $(\overline{xy})$ survives to time r. At our current valuation time (r years from issue), ${}_rL_{\overline{xy}}$ is conditioned on the contract being in force, that is, the last-survivor status $(\overline{xy})$ still exists at time r. Compared to the first-to-die insurance case where both insureds had to survived r years, here, there are three cases for which the policy is still in force:

- Case 1: both (x) and (y) are alive at time r, denoted by $\mathcal{E}_1(x,y,r) = \{T_x > r,\, T_y > r \,|\, T_{\overline{xy}} > r\}$ with $\Pr(\mathcal{E}_1(x,y,r)) = \frac{{}_rp_{xy}}{{}_rp_x + {}_rp_y - {}_rp_{xy}}$;
- Case 2: (x) is alive, while (y) is dead at time r, denoted by $\mathcal{E}_2(x,y,r) = \{T_x > r,\, T_y \le r \,|\, T_{\overline{xy}} > r\}$ with $\Pr(\mathcal{E}_2(x,y,r)) = \frac{{}_rp_x - {}_rp_{xy}}{{}_rp_x + {}_rp_y - {}_rp_{xy}}$;
- Case 3: (y) is alive, while (x) is dead at time r, denoted by $\mathcal{E}_3(x,y,r) = \{T_x \le r,\, T_y > r \,|\, T_{\overline{xy}} > r\}$ with $\Pr(\mathcal{E}_3(x,y,r)) = \frac{{}_rp_y - {}_rp_{xy}}{{}_rp_x + {}_rp_y - {}_rp_{xy}}$.

Let $K_{\overline{xy}}$ denote the the curtate-future-lifetime of last-survivor status $(\overline{xy})$. The prospective loss at time r is defined as the net difference between the present value of future benefits and the present value of future premiums. That is,

- Case 1

$$
{}_rL_{\overline{xy}} = \begin{cases} bv(K_{\overline{x+r:y+r}}+1) - \pi\ddot{a}(K_{\overline{x+r:y+r}}+1), & K_{\overline{x+r:y+r}} = 0, 1, \cdots, n-r+1, \\ cv(n-r) - \pi\ddot{a}(n-r), & K_{\overline{x+r:y+r}} = n-r, n-r+1, \cdots, \end{cases} \tag{5.1}
$$

- Case 2

$$
{}_rL_{\overline{xy}} = \begin{cases} bv(K_{x+r}+1) - \pi\ddot{a}(K_{x+r}+1), & K_{x+r} = 0, 1, \cdots, n-r+1, \\ cv(n-r) - \pi\ddot{a}(n-r), & K_{x+r} = n-r, n-r+1, \cdots, \end{cases}
$$

- Case 3

$$
{}_rL_{\overline{xy}} = \begin{cases} bv(K_{y+r}+1) - \pi\ddot{a}(K_{y+r}+1), & K_{y+r} = 0, 1, \cdots, n-r+1, \\ cv(n-r) - \pi\ddot{a}(n-r), & K_{y+r} = n-r, n-r+1, \cdots, \end{cases}
$$

where $K_{\overline{x+r:y+r}}$ is the curtate future lifetime of last-survivor status $(\overline{x+r:y+r})$. Given that case l (event $\mathcal{E}_l(x,y,r)$) is observed at time r, let $q^{(l)}(x,y,r,k)$ be the probability function that the status $(\overline{xy})$ survives $(k-1)$ years in the future and fail within time interval $(k, k+1]$, and $p^{(l)}(x, y, r, k)$ be the probability that the status $(\overline{xy})$ will survive to time k. Then, for $k = 0, 1, \cdots, n-1$, we have

$$q^{(l)}(x,y,r,k) = \begin{cases} {}_{k|}q_{\overline{x+r:y+r}}, & l = 1, \\ {}_{k|}q_{x+r}, & l = 2, \\ {}_{k|}q_{y+r}, & l = 3, \end{cases}$$

$$p^{(l)}(x,y,r,k) = \begin{cases} {}_{k}p_{\overline{x+r:y+r}}, & l = 1, \\ {}_{k}p_{x+r}, & l = 2, \\ {}_{k}p_{y+r}, & l = 3. \end{cases}$$

With the assumption that the interest rates follow an AR(1) process and the current rate at time r is δ_0, the β^{th} moment of the conditional prospective random variable ${}_rL_{\overline{xy}}$ corresponding to cases l, $l = 1, 2, 3$, given that the last-survivor status $(\overline{xy})$ survives to time r, can be calculated directly from the definition of ${}_rL_{\overline{xy}}$ given in (5.1) by conditioning on the random variable $K_{\overline{xy}}$ as follows:

$$\begin{aligned} E_{\delta_0}\Big[({}_rL_{\overline{xy}})^{\beta} \,|\, \mathcal{E}_l(x,y,r)\Big] &= E_{\delta_0}\Big[E[({}_rL_{\overline{xy}})^{\beta} \,|\, K_{\overline{xy}},\ \mathcal{E}_l(x,y,r)]\Big] \\ &= \sum_{k=0}^{n-r-1}\Big(b\,E_{\delta_0}[v(k+1)] - \pi\,E_{\delta_0}[\ddot{a}(k+1)]\Big)^{\beta} q^{(l)}(x,y,r,k) \\ &\quad + \Big(c\,E_{\delta_0}[v(n-r)] - \pi\,E_{\delta_0}[\ddot{a}(n-r)]\Big)^{\beta} p^{(l)}(x,y,r,n-r)\ . \end{aligned} \quad (5.2)$$

The conditional expectation of ${}_rL_{\overline{xy}}$ corresponding to cases l, $l = 1, 2, 3$, can be obtained by setting $\beta = 1$ in (5.2) as

$$\begin{aligned} E_{\delta_0}\left[{}_rL_{\overline{xy}} \,|\, \mathcal{E}_l(x,y,r)\right] &= \sum_{k=0}^{n-r-1}\Big(b\,E_{\delta_0}[v(k+1)] - \pi\,E_{\delta_0}[\ddot{a}(k+1)]\Big) q^{(l)}(x,y,r,k) \\ &\quad + \Big(c\,E_{\delta_0}[v(n-r)] - \pi\,E_{\delta_0}[\ddot{a}(n-r)]\Big) p^{(l)}(x,y,r,n-r)\ . \end{aligned} \quad (5.3)$$

With $\beta = 2$, we have the second moment of the prospective loss random variable corresponding to cases l, $l = 1, 2, 3$,

$$\begin{aligned}
&E_{\delta_0}\left[({}_rL_{\overline{xy}})^2 \,\middle|\, \mathcal{E}_l(x,y,r)\right] \\
&= \sum_{k=0}^{n-r-1} \Big(b^2\, E_{\delta_0}[v(k+1)^2] - 2b\pi\, E_{\delta_0}[v(k+1)\ddot{a}(k+1)] + \pi^2 E_{\delta_0}[\ddot{a}(k+1)^2]\Big) q^{(l)}(x,y,r,k) \\
&\quad + \Big(c^2\, E_{\delta_0}[v(n-r)] - 2c\pi\, E_{\delta_0}[v(n-r)\ddot{a}(n-r)] + \pi^2 E_{\delta_0}[\ddot{a}(n-r)^2]\Big) p^{(l)}(x,y,r,n-r)\ .
\end{aligned} \tag{5.4}$$

Then we can obtain the conditional expectation of ${}_rL_{\overline{xy}}$ by the law of total probability as a summation below,

$$E_{\delta_0}[({}_rL_{\overline{xy}})^\beta] = \sum_{l=1}^{3} E_{\delta_0}\left[({}_rL_{\overline{xy}})^\beta \,|\, \mathcal{E}_l(x,y,r)\right] \Pr(\mathcal{E}_l(x,y,r))\,, \tag{5.5}$$

where $\Pr(\mathcal{E}_l(x,y,r))$ can be found in the definition of $\mathcal{E}_l(x,y,r)$ at the beginning of this section.

5.2 First Expression for the Prospective Loss Variables for Non-homogeneous Portfolio

Now we investigate the property of the loss random variable for a joint last-to-die life insurance portfolio with m policies. At a given valuation date, consider a non-homogeneous portfolio with m joint last-to-die endowment life insurance policies. The terms of joint last-to-die contract n_i, death benefits b_i and pure endowment benefits c_i may vary from contract to contract for $i = 1, 2, \cdots, m$. The i^{th} joint last-to-die endowment insurance contract is issued to a pair aged (x_i, y_i) with premium π_i, death benefit b_i, and the pure endowment benefit c_i at time n_i. Let ${}_{r_i}L_{\overline{x_iy_i}}$ be the prospective loss random variable for the i^{th} joint last-to-die insurance contract, for $i = 1, 2, \cdots, m$. Note that ${}_{r_i}L_{\overline{x_iy_i}}$ is a conditional prospective loss random variable given that the last-survivor status $(\overline{x_iy_i})$ has survived r_i years and the contracts in the portfolio may have different survival time r_i's. Let L^{LSP} denote the loss random variable for the whole portfolio, i.e.,

$$L^{LSP} = \sum_{i=1}^{m} {}_{r_i}L_{\overline{x_iy_i}}\ ,$$

where ${}_{r_i}L_{\overline{x_iy_i}}$ is of the form

- Case 1

$$ {}_{r_i}L_{\overline{x_i y_i}} = \begin{cases} b_i v(K_{\overline{x_i+r_i:y_i+r_i}}+1) - \pi_i \ddot{a}(K_{\overline{x_i+r_i:y_i+r_i}}+1), \\ \qquad\qquad K_{\overline{x_i+r_i:y_i+r_i}} = 0, 1, \cdots, n_i - r_i - 1, \\ c_i v(n_i - r_i) - \pi_i \ddot{a}(n_i - r_i), \\ \qquad\qquad K_{\overline{x_i+r_i:y_i+r_i}} = n_i - r_i, n_i - r_i + 1, \cdots . \end{cases} $$

- Case 2

$$ {}_{r_i}L_{\overline{x_i y_i}} = \begin{cases} b_i v(K_{x_i+r_i}+1) - \pi_i \ddot{a}(K_{x_i+r_i}+1), & K_{x_i+r_i} = 0, 1, \cdots, n_i - r_i - 1, \\ c_i v(n_i - r_i) - \pi_i \ddot{a}(n_i - r_i), & K_{x_i+r_i} = n_i - r_i, n_i - r_i + 1, \cdots . \end{cases} $$

- Case 3

$$ {}_{r_i}L_{\overline{x_i y_i}} = \begin{cases} b_i v(K_{y_i+r_i}+1) - \pi_i \ddot{a}(K_{y_i+r_i}+1), & K_{y_i+r_i} = 0, 1, \cdots, n_i - r_i - 1, \\ c_i v(n_i - r_i) - \pi_i \ddot{a}(n_i - r_i), & K_{y_i+r_i} = n_i - r_i, n_i - r_i + 1, \cdots . \end{cases} $$

Note that the individual prospective loss random variables, ${}_{r_i}L_{\overline{x_i y_i}}$'s, in the portfolio are dependent because of the same interest rate process.

We make the assumption that the interest rates follow a conditional AR(1) process with initial interest rate δ_0 at our specific valuation time. By setting $\beta = 1$ in (5.5), the expression for $E_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}]$ is given by

$$ E_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}] = \sum_{l_i=1}^{3} E_{\delta_0}\left[{}_{r_i}L_{\overline{x_i y_i}} \,|\, \mathcal{E}_{l_i}(x_i, y_i, r_i)\right] Pr(\mathcal{E}_{l_i}(x_i, y_i, r_i)), \tag{5.6} $$

where the conditional expectation of the prospective loss random variable ${}_{r_i}L_{\overline{x_i y_i}}$ corresponding to case l_i can be obtained by replacing the set of parameters $\{x, y, r, n, b, c, l\}$ by $\{x_i, y_i, r_i, n_i, b_i, c_i, l_i\}$ in (5.3), for $l_i = 1, 2, 3$, as

$$ \begin{aligned} E_{\delta_0}\left[{}_{r_i}L_{\overline{x_i y_i}} \,|\, \mathcal{E}_{l_i}(x_i, y_i, r_i)\right] = & \sum_{k=0}^{n_i-r_i-1} \Big(b_i\, E_{\delta_0}[v(k+1)] - \pi_i\, E_{\delta_0}[\ddot{a}(k+1)]\Big) q^{(l_i)}(x_i, y_i, r_i, k_i) \\ & + \Big(c_i\, E_{\delta_0}[v(n_i - r_i)] - \pi_i\, E_{\delta_0}[\ddot{a}(n_i - r_i)]\Big) p^{(l_i)}(x_i, y_i, r_i, n_i - r_i) . \end{aligned} $$

Further we consider the conditional expectation of the product of two loss random variables ${}_{r_i}L_{\overline{x_i y_i}}$ and ${}_{r_j}L_{\overline{x_j y_j}}$ given that the last-survivor statuses $(\overline{x_i y_i})$ and $(\overline{x_j y_j})$ have survived r_i and r_j years, respectively.

The survival situations for two different pairs in the last-survivor statuses are complicated with 9 possibilities as shown in Table 5.1. By the law of total probability, we have

$$E_{\delta_0}[{}_{r_i}L_{\overline{x_iy_i}} \cdot {}_{r_j}L_{\overline{x_jy_j}}] = \sum_{l_i=1}^{3}\sum_{l_j=1}^{3} E_{\delta_0}\left[{}_{r_i}L_{\overline{x_iy_i}} \cdot {}_{r_j}L_{\overline{x_jy_j}} \,|\, \mathcal{E}_{l_i}(x_i,y_i,r_i),\ \mathcal{E}_{l_j}(x_j,y_j,r_j)\right] \cdot Pr(\mathcal{E}_{l_i}(x_i,y_i,r_i)) \cdot Pr(\mathcal{E}_{l_j}(x_j,y_j,r_j))\,, \tag{5.7}$$

where the conditional expectation of the product of two loss random variables corresponding to (l_i, l_j) within the 9 possibilities, $l_i = 1,2,3, l_j = 1,2,3$, can be derived as

$$\begin{aligned}
&E_{\delta_0}\left[{}_{r_i}L_{\overline{x_iy_i}} \cdot {}_{r_j}L_{\overline{x_jy_j}} \,|\, \mathcal{E}_{l_i}(x_i,y_i,r_i),\ \mathcal{E}_{l_j}(x_j,y_j,r_j)\right] \\
&= \sum_{k_i=0}^{n_i-r_i-1}\sum_{k_j=0}^{n_j-r_j-1}\left\{\begin{array}{c} b_ib_jE_{\delta_0}[v(k_i+1)v(k_j+1)] \\ +\pi_i\pi_jE_{\delta_0}[\ddot{a}(k_i+1)\ddot{a}(k_j+1)] \\ -b_i\pi_jE_{\delta_0}[v(k_i+1)\ddot{a}(k_j+1)] \\ -\pi_ib_jE_{\delta_0}[\ddot{a}(k_i+1)v(k_j+1)] \end{array}\right\} q^{(l_i)}(x_i,y_i,r_i,k_i)\cdot q^{(l_j)}(x_j,y_j,r_j,k_j) \\
&\quad + \sum_{k_i=0}^{n_i-r_i-1}\left\{\begin{array}{c} b_ic_jE_{\delta_0}[v(k_i+1)v(n_j-r_j)] \\ +\pi_i\pi_jE_{\delta_0}[\ddot{a}(k_i+1)\ddot{a}(n_j-r_j)] \\ -b_i\pi_jE_{\delta_0}[v(k_i+1)\ddot{a}(n_j-r_j)] \\ -\pi_ic_jE_{\delta_0}[\ddot{a}(k_i+1)v(n_j-r_j)] \end{array}\right\} q^{(l_i)}(x_i,y_i,r_i,k_i)\cdot p^{(l_j)}(x_j,y_j,r_j,n_j-r_j) \\
&\quad + \sum_{k_j=0}^{n_j-r_j-1}\left\{\begin{array}{c} b_jc_iE_{\delta_0}[v(k_j+1)v(n_i-r_i)] \\ +\pi_j\pi_iE_{\delta_0}[\ddot{a}(k_j+1)\ddot{a}(n_i-r_i)] \\ -b_j\pi_iE_{\delta_0}[v(k_j+1)\ddot{a}(n_i-r_i)] \\ -\pi_jc_iE_{\delta_0}[\ddot{a}(k_j+1)v(n_i-r_i)] \end{array}\right\} p^{(l_i)}(x_i,y_i,r_i,n_i-r_i)\cdot q^{(l_j)}(x_j,y_j,r_j,k_j) \\
&\quad + \left\{\begin{array}{c} c_ic_jE_{\delta_0}[v(n_i-r_i)v(n_j-r_j)] \\ +\pi_i\pi_jE_{\delta_0}[\ddot{a}(n_i-r_i)\ddot{a}(n_j-r_j)] \\ -c_i\pi_jE_{\delta_0}[v(n_i-r_i)\ddot{a}(n_j-r_j)] \\ -\pi_ic_jE_{\delta_0}[\ddot{a}(n_i-r_i)v(n_j-r_j)] \end{array}\right\} p^{(l_i)}(x_i,y_i,r_i,n_i-r_i)\cdot p^{(l_j)}(x_j,y_j,r_j,n_j-r_j)\,.
\end{aligned}$$

Similarly, the second moment can be obtained from (5.5) by setting $\beta = 2$, that is,

$$E_{\delta_0}[({}_{r_i}L_{\overline{x_iy_i}})^2] = \sum_{l_i=1}^{3} E_{\delta_0}\left[({}_{r_i}L_{\overline{x_iy_i}})^2 \,|\, \mathcal{E}_{l_i}(x_i,y_i,r_i)\right] Pr(\mathcal{E}_{l_i}(x_i,y_i,r_i))\,, \tag{5.8}$$

where the conditional second moment of the prospective loss random variable ${}_{r_i}L_{\overline{x_iy_i}}$ corresponding to cases l_i, $l_i = 1,2,3$, can be obtained by replacing the set of parameters

Table 5.1: 9 Possible cases when $(\overline{x_i y_i})$ and $(\overline{x_j y_j})$ do not fail at valuation

(l_i, l_j)	x_i	y_i	x_j	y_j
$(1,1)$	alive	alive	alive	alive
$(1,2)$	alive	alive	alive	dead
$(1,3)$	alive	alive	dead	alive
$(2,1)$	alive	dead	alive	alive
$(2,2)$	alive	dead	alive	dead
$(2,3)$	alive	dead	dead	alive
$(3,1)$	dead	alive	alive	alive
$(3,2)$	dead	alive	alive	dead
$(3,3)$	dead	alive	dead	alive

$\{x, y, r, b, c, n, l\}$ by $\{x_i, y_i, r_i, b_i, c_i, n_i, l_i\}$ in (5.4) as

$$\begin{aligned}
&E_{\delta_0}\left[({}_{r_i}L_{\overline{x_i y_i}})^2 \,|\, \mathcal{E}_{l_i}(x_i, y_i, r_i)\right] \\
&= \sum_{k=0}^{n_i - r_I - 1} \Big(b_i^2\, E_{\delta_0}[v(k_i+1)^2] - 2b_i\pi_i\, E_{\delta_0}[v(k_i+1)\ddot{a}(k_i+1)] \\
&\quad + \pi_i^2 E_{\delta_0}[\ddot{a}(k_i+1)^2]\Big)\, q^{(l)}(x_i, y_i, r_i, k_i) + \Big(c_i^2\, E_{\delta_0}[v(n_i - r_i)] \\
&\quad - 2c_i\pi_i\, E_{\delta_0}[v(n_i - r_i)\ddot{a}(n_i - r_i)] + \pi_i^2 E_{\delta_0}[\ddot{a}(n_i - r_i)^2]\Big) p^{(l_i)}(x_i, y_i, r_i, n_i - r_i)\ .
\end{aligned}$$

Using (5.6)-(5.8), expressions for $Var_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}]$ and $Cov_{\delta_0}\left[{}_{r_i}L_{\overline{x_i y_i}},\, {}_{r_j}L_{\overline{x_j y_j}}\right]$ can be derived from

$$Var_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}] = E_{\delta_0}[({}_{r_i}L_{\overline{x_i y_i}})^2] - E_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}]\ , \tag{5.9}$$

$$Cov_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}},\, {}_{r_j}L_{\overline{x_j y_j}}] = E_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}} \cdot {}_{r_j}L_{\overline{x_j y_j}}] - E_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}] \cdot E[{}_{r_j}L_{\overline{x_j y_j}}]\ . \tag{5.10}$$

Therefore, the expression for $E\left[L^{LSP}\right]$ is derived by summing the conditional expectations of loss random variables, $E_{\delta_0}[{}_{r_i}L_{\overline{x_i y_i}}]$, given in (5.6), for all policies, as

$$E_{\delta_0}\left[L^{LSP}\right] = E_{\delta_0}\left[\sum_{i=1}^{m} {}_{r_i}L_{\overline{x_i y_i}}\right] = \sum_{i=1}^{m} E_{\delta_0}\left[{}_{r_i}L_{\overline{x_i y_i}}\right]\ .$$

The expression for $Var[L^{LSP}]$ is determined by

$$\begin{aligned} Var_{\delta_0}\left[L^{LSP}\right] &= Var_{\delta_0}\left[\sum_{i=1}^{m} {}_{r_i}L_{\overline{x_iy_i}}\right] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} Cov_{\delta_0}\left[{}_{r_i}L_{\overline{x_iy_i}},\ {}_{r_j}L_{\overline{x_jy_j}}\right] \\ &= \sum_{i=1}^{m} Var_{\delta_0}\left[{}_{r_i}L_{\overline{x_iy_i}}\right] + \sum_{i=1}^{m}\sum_{j=1, j\neq i}^{m} Cov_{\delta_0}\left[{}_{r_i}L_{\overline{x_iy_i}},\ {}_{r_j}L_{\overline{x_jy_j}}\right] , \end{aligned}$$

where the corresponding variance and covariance expressions are given in (5.9) and (5.10).

5.3 Second Expression for the Prospective Loss Variables for Non-homogeneous Portfolio

Similar to the idea described in Section 4.3, we study the loss random variable for a portfolio in the joint last-to-die life insurance case alternatively by focusing on the future annual payouts of the whole portfolio. For the same non-homogeneous portfolio studied in the previous section, following Parker (1997) and Marceau and Gaillardetz (1999), let $\overline{CF}(j)$ be the random cash flow payable at time j with respect to the whole joint last-to-die endowment insurance portfolio. Similarly, we define the latest possible cash flow occurrence time in the portfolio: $n = \max_{1\leq i\leq n}(n_i - r_i)$. The interpretation of $\overline{CF}(j)$ is the net difference between benefits paid and premiums received at time j, $j = 1, 2, \cdots, n$, namely,

$$\overline{CF}(j) = \sum_{i=1}^{m} \overline{D}_{i,j}\, b_i\, 1_{(n_i - r_i \geq j)} - \sum_{i=1}^{m} \overline{S}_{i,j}\, \pi_i\, 1_{(n_i - r_i > j)} + \sum_{i=1}^{m} \overline{S}_{i,n_i - r_i}\, c_i\, 1_{(n_i - r_i = j)} , \qquad (5.11)$$

where the random variables $\overline{D}_{i,j}$ and $\overline{S}_{i,j}$ are defined by

$$\overline{D}_{i,j} = \begin{cases} 1, & \text{if the second death in policy } i \text{ occurs within the interval } (j-1, j]\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

$$\overline{S}_{i,j} = \begin{cases} 1, & \text{if any insured in policy } i \text{ survives to time } j\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

For $j = 0$, the expression is

$$\overline{CF}(0) = -\sum_{i=1}^{m} \overline{S}_{i,j}\, \pi_i\, 1_{(n_i - r_i > 0)} \,.$$

Note that $D_{i,0} = 0$ for any i. Then, for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, we have

$$E[\overline{D}_{i,j}] = \sum_{l_i=1}^{3} q^{(l_i)}(x_i, y_i, r_i, j-1) \cdot Pr(\mathcal{E}_{l_i}(x_i, y_i, r_i)) \,, \tag{5.12}$$

$$E[\overline{S}_{i,j}] = \sum_{l_i=1}^{3} p^{(l_i)}(x_i, y_i, r_i, j) \cdot Pr(\mathcal{E}_{l_i}(x_i, y_i, r_i)) \,, \tag{5.13}$$

$$Var[\overline{D}_{i,j}] = E[\overline{D}_{i,j}] - \left(E[\overline{D}_{i,j}]\right)^2 \,, \tag{5.14}$$

$$Var[\overline{S}_{i,j}] = E[\overline{S}_{i,j}] - \left(E[\overline{S}_{i,j}]\right)^2 \,. \tag{5.15}$$

For $0 \le k < j$ and $i = 1, 2, \cdots, m$, we have the following covariances,

$$Cov[\overline{D}_{i,k}, \overline{D}_{i,j}] = -E[\overline{D}_{i,k}] \cdot E[\overline{D}_{i,j}] \,, \tag{5.16}$$

$$Cov[\overline{D}_{i,j}, \overline{S}_{i,j}] = -E[\overline{D}_{i,j}] \cdot E[\overline{S}_{i,j}] \,, \tag{5.17}$$

$$Cov[\overline{D}_{i,k}, \overline{S}_{i,j}] = -E[\overline{D}_{i,k}] \cdot E[\overline{S}_{i,j}] \,, \tag{5.18}$$

$$Cov[\overline{D}_{i,j}, \overline{S}_{i,k}] = E[\overline{D}_{i,j}] - E[\overline{D}_{i,j}] \cdot E[\overline{S}_{i,k}] \,, \tag{5.19}$$

$$Cov[\overline{S}_{i,j}, \overline{S}_{i,k}] = E[\overline{S}_{i,j}] - E[\overline{S}_{i,j}] \cdot E[\overline{S}_{i,k}] \,. \tag{5.20}$$

Based on the assumption in Section 3.3, the random variables $\{K_{\overline{x_i+r_i:y_i+r_i}};\ i = 1, 2, \cdots\}$ are mutually independent. Therefore, all the covariances above are zero for a pair of variables from two different policies. For instance, considering two policies i_1 and i_2 with $i_1 \neq i_2$, we have $Cov[\overline{D}_{i_1,k}, \overline{D}_{i_2,j}] = 0$ for any k and j.

Similar (4.20), the relationship between the first expression of the prospective loss variables studied in Section 5.2 and the one studied in this section for a non-homogeneous portfolio is

$$L^{LSP} = \sum_{j=0}^{n} \overline{CF}(j) v(j) = \sum_{i=1}^{m} {}_{r_i}L_{\overline{x_i y_i}} \,.$$

By taking the expectations of both sides of (5.11) and using (5.12) and (5.13), we obtain, for $j = 1, 2, \cdots, n$,

$$\begin{aligned} E\left[\overline{CF}(j)\right] &= \sum_{i=1}^{m} \left[\left(E[\overline{D}_{i,j}]\, b_i\, 1_{(n_i - r_i \ge j)} - E[\overline{S}_{i,j}]\, \pi_i\, 1_{(n_i - r_i > j)} + E[\overline{S}_{i,n_i - r_i}]\, c_i\, 1_{(n_i - r_i = j)}\right)\right] \\ &= \sum_{i=1}^{m} \sum_{l_i=1}^{3} \Big(q^{(l_i)}(x_i, y_i, r_i, j-1)\, b_i\, 1_{(n_i - r_i \ge j)} - p^{(l_i)}(x_i, y_i, r_i, j)\, \pi_i\, 1_{(n_i - r_i > j)} \\ &\qquad + p^{(l_i)}(x_i, y_i, r_i, n_i - r_i)\, c_i\, 1_{(n_i - r_i = j)} \Big) \cdot Pr(\mathcal{E}_{l_i}(x_i, y_i, r_i)) \,, \end{aligned} \tag{5.21}$$

and for $j = 0$, the expression is

$$E[\overline{CF}(0)] = \overline{CF}(0) = -\sum_{i=1}^{m} \pi_i.$$

Moreover, for this non-homogeneous last-to-die life insurance portfolio, the expressions for $E\left[\overline{CF}(j)\right]$ and $Cov\left[\overline{CF}(k), \overline{CF}(j)\right]$ for $j < k$, analogous to (4.22) and (4.23), can be obtained by replacing the variance and covariance of $\{D_{i,j}\}$ and $\{S_{i,j}\}$ by the corresponding set of equations given in (5.14)-(5.20) for $\{\overline{D}_{i,j}\}$ and $\{\overline{S}_{i,j}\}$.

Similar to (4.24), we have the following expression for $E_{\delta_0}\left[L^{LSP}\right]$:

$$E_{\delta_0}\left[L^{LSP}\right] = E\left[\sum_{j=0}^{n} \overline{CF}(j)v(j) \,\Big|\, \delta(0) = \delta_0\right] = \sum_{j=0}^{n} E[\overline{CF}(j)]E_{\delta_0}[v(j)]\,,$$

and two expressions for the conditional variance, $Var_{\delta_0}\left[L^{LSP}\right]$ corresponding to (4.25) and (4.28):

$$Var_{\delta_0}\left[L^{LSP}\right] = E\left[Var\left[L^{LSP} \,\Big|\, \{K_{\overline{x_i y_i}}\},\, \delta(0) = \delta_0\right]\right] + Var\left[E\left[L^{LSP} \,\Big|\, \{K_{\overline{x_i y_i}}\},\, \delta(0) = \delta_0\right]\right], \tag{5.22}$$

and

$$Var\left[L^{LSP}\right] = E\left[Var\left[L^P \,\Big|\, \{\delta(k)\}\right]\right] + Var\left[E\left[L^P \,\Big|\, \{\delta(k)\}\right]\right]. \tag{5.23}$$

Parallel to (4.25) and (4.28), the terms in (5.22) and (5.23) can be obtained by replacing $CF(j)$ by $\overline{CF}(j)$ in (4.26)-(4.27) and (4.29)-(4.30). Similar to the argument we made in Section 4.3, we can decompose the total riskiness into the insurance risk and the investment risk. The first term of (5.22) represents the investment risk while the second term represents the insurance risk. In the alternative decomposition expression of the total riskiness given in (5.23), the first term measures the insurance risk and the second term measures the investment risk.

Chapter 6

Numerical Illustration

We illustrate only the joint first-to-die life insurance case for which the valuation formulas for a single policy and a non-homogenous portfolio are presented. In this chapter, we analyze the impact of different mortality assumptions and interest rate assumptions on the expectations and variances of the prospective loss variables for both the single policy and the portfolio. The valuation of the joint last-to-die life insurance can be done similarly.

6.1 Single Policy

Consider an n-year term/endowment single first-to-die policy issued to a pair at the age of (60,50) with \$1 death benefit and \$1 pure endowment benefit. The level premium is calculated at issue according to the equivalence principle. We value this single contract at time r with $r < n$. The first two moments of the prospective loss variables for this single policy are calculated under the copula mortality assumption and an AR(1) process as the model for the rate of return. The parameters of the copula mortality model are given in Table 3.2. There are four parameters, δ, δ_0, ϕ and σ, in AR(1) process for the rate of return, in which δ is the long term mean for the force of interest rate process, δ_0 is the starting value, ϕ controls the speed of convergence from the starting value to the long term mean, and σ determines the volatility of the process. The parameters for AR(1) process used in this project are $\delta = 0.06$, $\delta_0 = 0.08$, $\phi = 0.9$ and $\sigma = 0.01$. In Tables 6.2-6.4, δ_0, ϕ and σ are changed respectively to test the sensitivity of loss random variable to the choice of parameters.

We begin by analyzing the performance of the prospective loss random variable for a single policy at different valuation times r. In actuarial terminology, the expected value of the prospective loss random variable is the benefit reserve. The reserve is a liability that should be recognized in the financial statement of an insurance company. The standard deviation of the prospective loss random variable describes the possible volatility of the liability that should be paid off to the policyholders in the future. Frees et.al (1996) focused on the annuity valuation with fixed interest rate. We apply his mortality assumptions to joint first-to-die life insurance in the environment of stochastic interest rates.

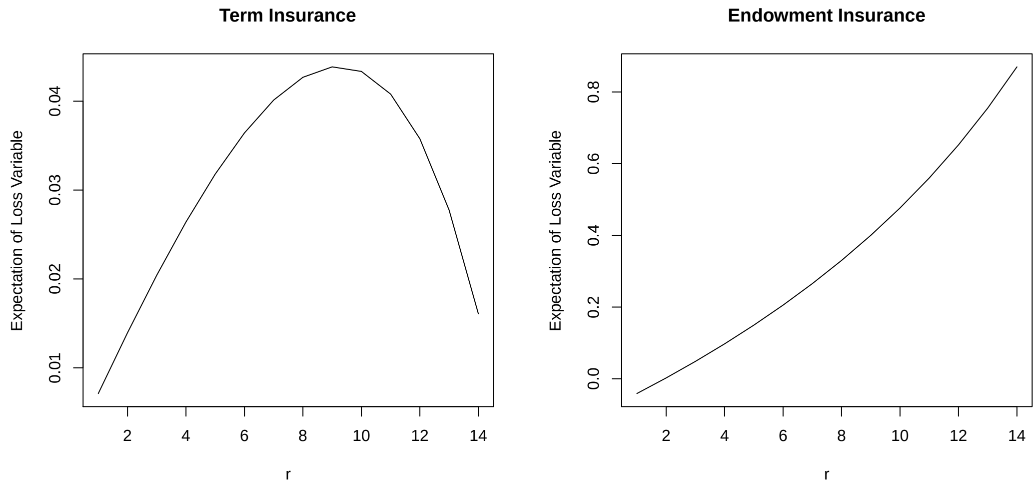


Figure 6.1: Expected values of prospective loss for a 15-year term and endowment contract

Figure 6.1 plots the expectation of the prospective loss random variable for a 15-year term and endowment single policy with r. We observe that the expected value of the prospective loss variables for a term insurance first increases then decreases. This is consistent with the pattern of the cash flows of term insurance: cash flows are negative in the very early years when a small reserve is gradually built up, then cash flows become positive and the reserve is used to pay death benefits in the later years. In the case of endowment insurance, the expected value of the prospective loss variable increases all the way with r increasing. This is due to the fact that reserves are built up gradually for a pure endowment paid at the end of year 15 if the contract is still in force at the beginning of that year.

Figure 6.2 shows us the total riskiness in terms of the standard deviation of the prospective loss random variable for the same single policy as in Figure 6.1 with r. In order to gain an understanding of this figure, we need to recall the decomposition of the total riskiness

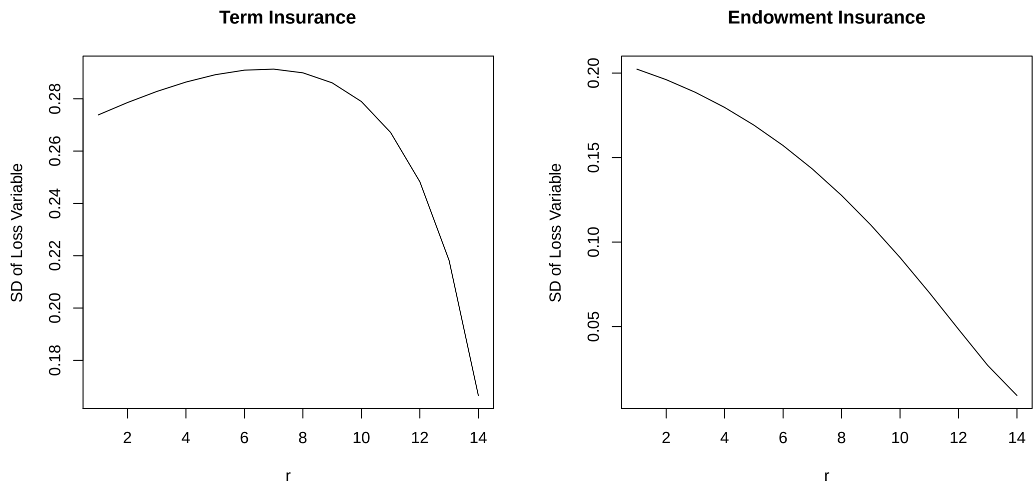


Figure 6.2: Standard Deviation of prospective loss for a 15-year term and endowment contract

Table 6.1: Risk composition for 15-year term/endwoment life insurance at year r

	$r = 1$	$r = 3$	$r = 5$	$r = 7$	$r = 9$	$r = 11$	$r = 13$
Term							
Insurance Risk	.074952	.079941	.083608	.084856	.081837	.071338	.047605
Investment Risk	.000031	.000029	.000024	.000016	.000009	.000003	.000001
Endowment							
Insurance Risk	.033037	.028314	.022389	.015659	.008906	.003347	.000361
Investment Risk	.002556	.002730	.002702	.002411	.001835	.001053	.000315

of the contract. As discussed in Chapter 4, the variance of prospective loss variable can be decomposed into the insurance risk and the investment risk. The insurance risk given in (4.29) can be considered as discounted mortality risks where the financial risk can be averaged out in an ideal case. For the 15-year term insurance, the standard deviation of the prospective loss random variable increases slightly then decreases. We note that the decrease and increase are due to the balance of the decline of the insurance risk and the increase of the investment risk. As shown in Figure 6.3 and Table 6.1, the insurance risk increases first then decreases with larger values of r and dominates the overall variability of the prospective loss random variable. In the case of the endowment insurance, the insurance risk decreases with larger values of r, and dominates the total riskiness though the proportion of the insurance risk to the total risk $E[Var[L^P \mid \{\delta(k)\}]]/Var[L^P]$ is decreasing over the time period.

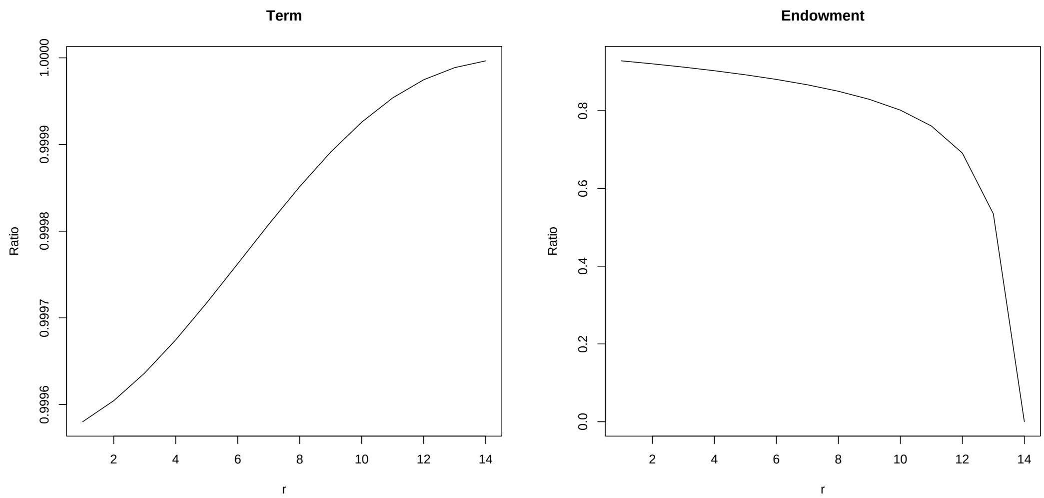


Figure 6.3: Proportion of insurance risk to total risk for a 15-year term and endowment contract

Table 6.2: Scenario 1: Single 5-year term/endowment life insurance contract valued at year r under copula assumption

δ_0	Term: $E[{}_rL_{60:50}]$, $SD[{}_rL_{60:50}]$				Endowment: $E[{}_rL_{60:50}]$, $SD[{}_rL_{60:50}]$			
	$r=1$	$r=2$	$r=3$	$r=4$	$r=1$	$r=2$	$r=3$	$r=4$
0.04	0.00192	0.00302	0.00318	0.00223	0.18161	0.37204	0.57167	0.78088
	0.18566	0.16892	0.14494	0.10773	0.08054	0.05385	0.02902	0.00959
0.06	0.00185	0.00293	0.00310	0.00219	0.17375	0.35935	0.55799	0.77103
	0.17830	0.16335	0.14121	0.10581	0.07931	0.05351	0.02900	0.00942
0.08	0.00178	0.00284	0.00302	0.00214	0.16624	0.34709	0.54460	0.76125
	0.17133	0.15802	0.13760	0.10392	0.07812	0.05319	0.02898	0.00925

Other parameter of the AR(1) model: $\delta = 0.06$, $\phi = 0.9$, $\sigma = 0.01$

We now consider three scenarios to test the sensitivity of the prospective loss random variables to the parameters of that AR(1) process. In the first scenario, we have three possible realizations of δ_0 (4%, 6% and 8%), and the other parameters are fixed at $\delta = 0.06$, $\phi = 0.9$ and $\sigma = 0.01$. From Table 6.2, we can see that the expectations (first lines) and the standard deviations (second lines) of the prospective loss variables under the copula mortality assumption decrease for both term and endowment insurance when the starting value δ_0 increases. This is due to the fact that the cash flows are discounted at a lower rate of return.

Table 6.3: Scenario 2: Single 5-year term/endowment life insurance contract valued at year r under copula assumption

ϕ	Term: $E[{}_rL_{60:50}]$, $SD[{}_rL_{60:50}]$				Endowment: $E[{}_rL_{60:50}]$, $SD[{}_rL_{60:50}]$			
	$r=1$	$r=2$	$r=3$	$r=4$	$r=1$	$r=2$	$r=3$	$r=4$
0.5	0.00180	0.00285	0.00301	0.00211	0.17020	0.35261	0.54920	0.76314
	0.17552	0.16103	0.13946	0.10476	0.07508	0.05040	0.02729	0.00932
0.7	0.00178	0.00282	0.00299	0.00211	0.16786	0.34899	0.54572	0.76123
	0.17373	0.15968	0.13858	0.10434	0.07616	0.05160	0.02813	0.00929
0.9	0.00178	0.00284	0.00302	0.00214	0.16624	0.34709	0.54460	0.76125
	0.17133	0.15802	0.13760	0.10392	0.07812	0.05319	0.02898	0.00925

Other parameter of the AR(1) model: $\delta = 0.06$, $\delta_0 = 0.08$, $\sigma = 0.01$

Table 6.4: Scenario 3: Single 5-year term/endowment life insurance contract valued at year r under copula assumption

σ	Term: $E[{}_rL_{60:50}]$, $SD[{}_rL_{60:50}]$				Endowment: $E[{}_rL_{60:50}]$, $SD[{}_rL_{60:50}]$			
	$r=1$	$r=2$	$r=3$	$r=4$	$r=1$	$r=2$	$r=3$	$r=4$
0.01	0.00178	0.00284	0.00302	0.00214	0.16624	0.34709	0.54460	0.76125
	0.17133	0.15802	0.13760	0.10392	0.07812	0.05319	0.02898	0.00925
0.02	0.00177	0.00282	0.00300	0.00213	0.16572	0.34622	0.54367	0.76061
	0.17183	0.15829	0.13772	0.10396	0.09204	0.06693	0.04137	0.01850
0.03	0.00175	0.00279	0.00297	0.00211	0.16483	0.34475	0.54209	0.75952
	0.17268	0.15874	0.13791	0.10401	0.11193	0.08523	0.05630	0.02777

Other parameter of the AR(1) model: $\delta = 0.06$, $\delta_0 = 0.08$, $\phi = 0.09$

In the second scenario, we consider three values of ϕ (0.9, 0.7 and 0.5) and the other parameters are fixed at $\delta = 0.06$, $\delta_0 = 0.08$ and $\sigma = 0.01$. As shown in Table 6.3, for a term insurance, with increasing ϕ, the expectations of the loss random variable first decrease then increase. The case for endowment is simpler; the expectation of the prospective loss random variable increases with lager values of ϕ.

In the third scenario, we consider three values of σ (0.01, 0.02 and 0.03) while keeping the other parameters fixed at $\delta = 0.06$, $\delta_0 = 0.08$ and $\phi = 0.9$. Expectations of the prospective loss variables for both types of insurance decrease with the increase of σ due to that cash flows are discounted at a lower rate of return. And standard deviations the prospective loss variables for both types of insurance increase with the increase of σ. Overall, standard deviations of the prospective loss random variable are relatively sensitive to the choice of parameters of the AR(1) process compared to the corresponding expectations; among all four parameters in the rate of return model, the effect of ϕ is the most significant one except the standard deviation of endowment life insurance.

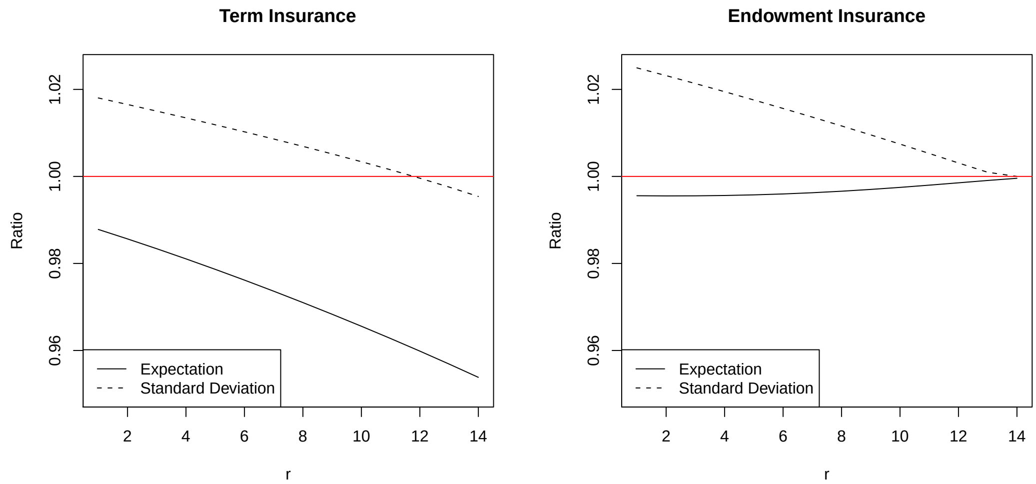


Figure 6.4: Ratio of dependent to independent expectation and standard deviation of prospective loss random variable for 15-year term/endowment life insurance issued to a pair (60,50)

Figure 6.4 displays the effect of mortality dependence on prospective loss variables. In the case of the 15-year term insurance issued to (60,50), the ratio of dependent to independent standard deviations of the prospective loss random variable is consistent with the ratio of conditional probability that (60,50) fails in year k for 15 years (see Figure 6.5), since the insurance uncertainty increases with larger values of the death probability. In the case of the endowment insurance, the ratio of dependent to independent standard deviations of the prospective loss random variable decreases slower and reaches 1 since the uncertainty of survival near the end of 15 years dominates the insurance risk.

To assess the effect of contract initiation ages, Figure 6.6 presents three-dimension plots of the ratio of the reserve for a single first-to-die life insurance by male (x) ages and female (y) ages, where both (x) and (y) change from 50 to 80. For the term insurance, the surface is

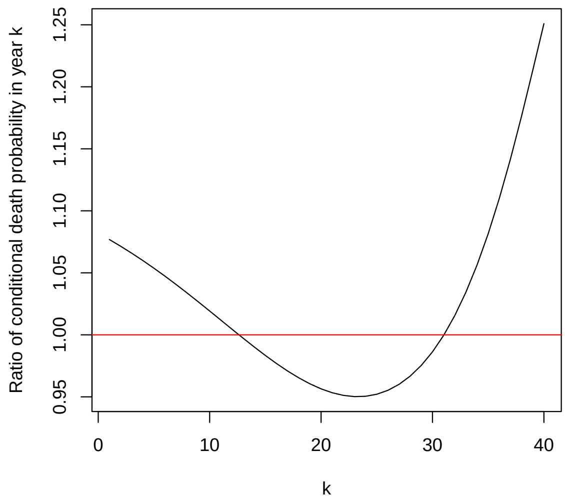


Figure 6.5: Ratio of conditional probability that (60,50) die in year k under Frank's copula to those under independent assumption

consistent with the three-dimensional plot of the ratio of dependent to independent annuity values in Frees et al. (1996) for the same range of ages. This is due to the dominance of the immediate annuity part in the definition of the loss random variable. Compared to Frees et al. (1996), the left surface in Figure 6.6 is less symmetric in x and y, since the difference between two marginal distributions has been magnified for term insurance. Further, there is an interaction effect of x and y on the ratios. This indicates that the ratio is larger for larger values of either x or y when compared to the larger values of both x and y which is consistent with the results in Frees et al. (1996). In the case of endowment insurance, the surface is flat and below 1. This can be explained by the fact that the increase of death probabilities is offset by the decline in the survival probabilities, while the premium is not dominant compared to the pure endowment payment.

Figure 6.7 displays the ratio of the standard deviation of the loss random variable under dependent assumption to the standard deviation under independent mortality assumption for a 10-year term or endowment life insurance contracts valued at time 5. Obviously, there exists an interaction between the male age (x) and the female age (y) for both the term insurance case and the endowment insurance case. The surface for term insurance is flatter than in the endowment case. This is due to the mortality risk related to the pure endowment

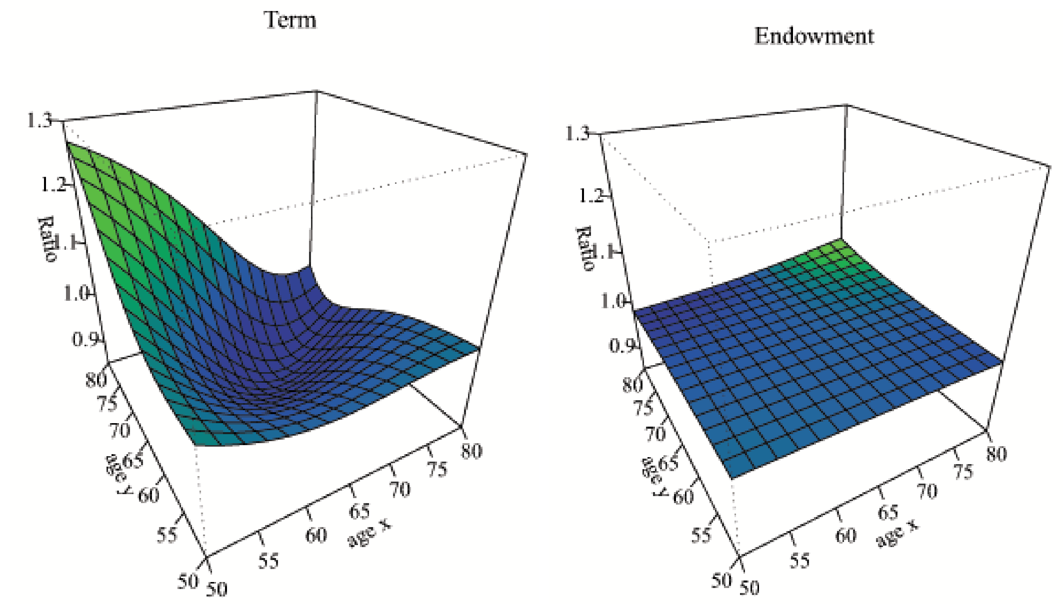


Figure 6.6: Ratio of expectation of loss random variable under copula mortality assumption to those under independent mortality assumption for a 10-year term/endowment life insurance contracts valued at time 5

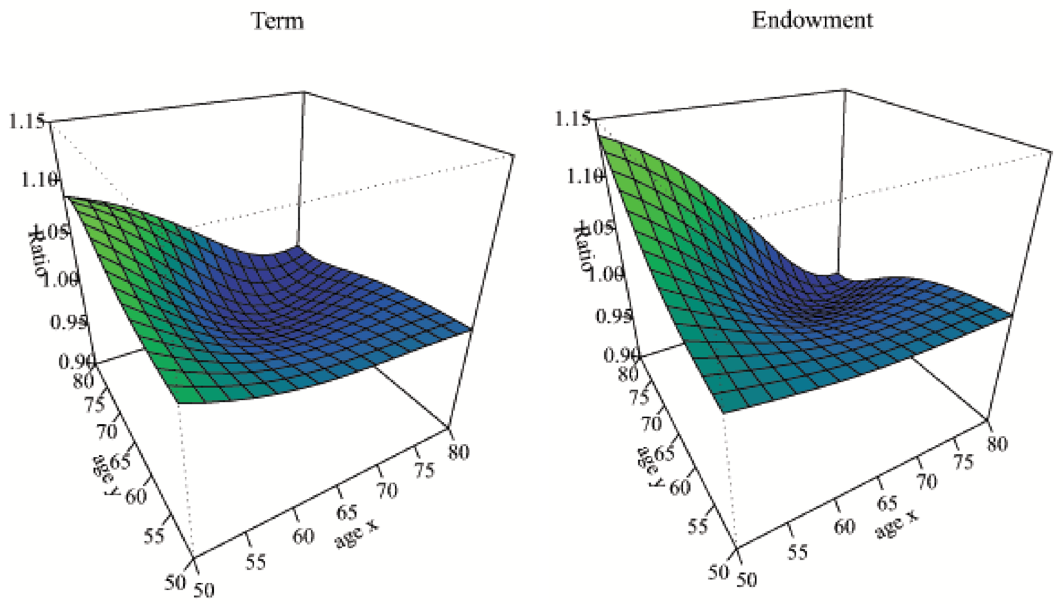


Figure 6.7: Ratio of standard deviation of loss random variable under copula mortality assumption to those under independent mortality assumption for a 10-year term/endowment life insurance contracts valued at time 5

payment which is not included in the term insurance.

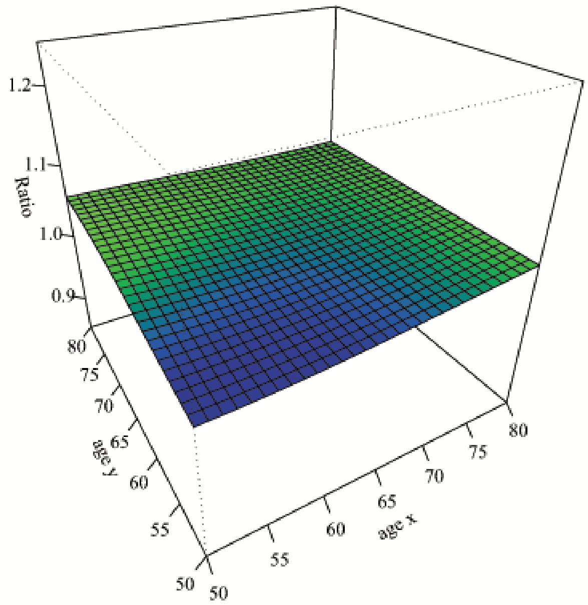


Figure 6.8: Ratio of expectation of loss random variable under common shock assumption to those under independent mortality assumption for a 10-year term life insurance contracts valued at time 5

As we mentioned in Chapter 3, common shock is another popular assumption for dependent mortality structure. Figure 6.8 displays the ratio of the expectation of loss random variable under the common shock assumption to the expectation under the independent mortality assumption for a 10-year term life insurance contracts valued at time 5. The surface is flat with ratios around 1.03. It is obvious that there is no interaction between the male age (x) and the female (y). We note that the common shock model does not seem to take into account all the dependencies that we observe in the industry data which is consistent to the results in Frees et al. (1996). Since the dependence between lives cannot be detected by the common shock model, we do not further discuss the performance of the loss random variable under this model.

6.2 Portfolio

After analyzing the properties of a single policy, we continue to investigate the properties of a portfolio of first-to-die insurance. An important risk management tool is the pooling of the risks. The basic idea is to reduce the total risk by diversification of some uncertainties through pooling. Consider a homogenous portfolio of first-to-die endowment insurance issued to pairs at the age of (60,50) with $1 death benefit and $1 pure endowment benefit. The level premium is calculated according to the equivalence principle. The expectation of the prospective loss variables for each policy in the portfolio is the same as the expectation for a single policy discussed in the last section.

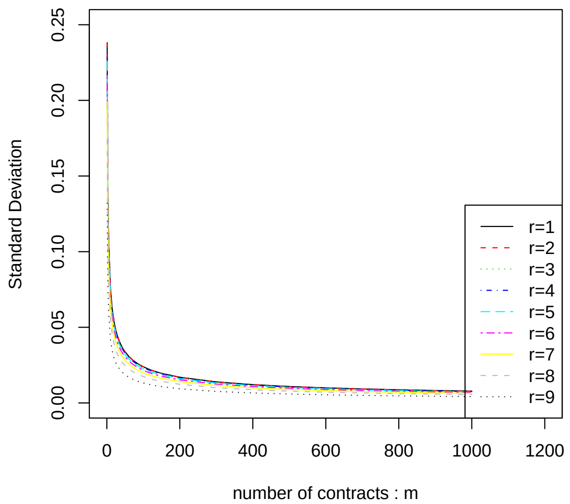


Figure 6.9: Standard deviation of portfolio of m 10-year term life insurance contracts valued at year r ($r = 1, 2, \cdots, 9$) under copula mortality assumption

Figure 6.9 plots the standard deviation of the prospective loss random variable per policy in a portfolio of size m, $SD\left[{}_rL^P/m\right]$. The nine curves represent nine different valuation times before the policy expires. As expected, the standard deviation decreases quickly as the size of the portfolio grows at all valuation times. The reason for the decrease is the diversification of the mortality risk (insurance risk) through pooling. Comparing Figures 6.9 and 6.10, we notice that pooling is more effective in the term insurance case than in the endowment insurance case. For instance, for a 10-year life insurance valued at time 1,

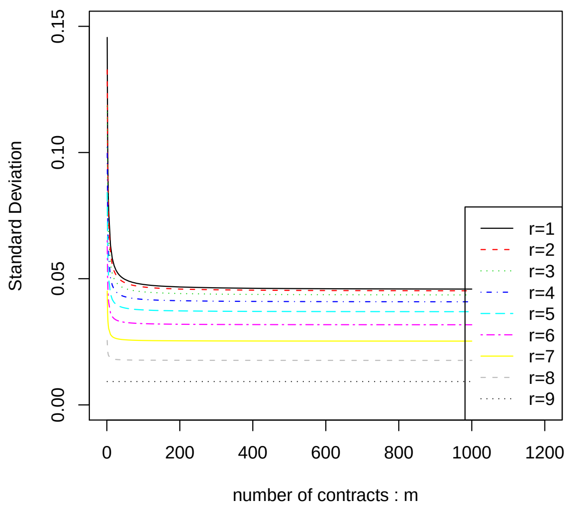


Figure 6.10: Standard deviation of portfolio of m 10-year endowment life insurance contracts valued at year r under copula assumption

with the growth of the size of portfolio from $m = 1$ to $m = 1000$, the standard deviation per policy for term insurance declines by 96.7% (from 0.238 to 0.008), while the decrease for endowment insurance is 68.5% (from 0.146 to 0.005).

The numerical results shown in Table 6.5 provide an insight into the sensitivity of the standard deviations, $SD\left[{}_rL^P/m\right]$, to the choice of parameters for the AR(1) process as the portfolio size m changes. We observe that the convergence speed of $SD\left[{}_rL^P/m\right]$ is not quite sensitive to the choice of parameters since the investment risk can not be diversified by the pooling. However, compared to the corresponding single policy case, as the size of the portfolio m goes to infinity, the limit of $SD\left[{}_rL^P/m\right]$ is more sensitive to the choice of parameters in AR(1) process.

Figure 6.11 presents the effect of mortality dependence on the total riskiness of the portfolio. The ratios of $SD\left[{}_rL^P/m\right]$ under the dependent mortality assumption to that under the independent mortality assumption are displayed with the growth of the portfolio size m. In the case of the term insurance, the ratio (solid line) slightly decreases with larger portfolio sizes and stays above one, implying that $SD\left[{}_rL^P/m\right]$ values under the dependent mortality assumption are larger than those under the independent mortality assumption. This is due to the growing insurance (mortality) uncertainty under the dependent mortality

Table 6.5: Portfolio of m 5-year term life insurance contracts valued at year 1 under copula assumption

			$SD[{}_1L_{60:50}/m]$				
δ_0	ϕ	σ	$m=1$	$m=100$	$m=10000$	$m=1000000$	$m\to\infty$
0.04	0.9	0.01	0.185658	0.018574	0.00194	0.000592	0.000562
		0.03	0.187208	0.018797	0.002531	0.001713	0.001703
	0.5	0.01	0.180946	0.018098	0.001844	0.000399	0.000355
		0.03	0.181674	0.018199	0.002108	0.001085	0.00107
0.06	0.9	0.01	0.178301	0.017838	0.001862	0.000567	0.000538
		0.03	0.179749	0.018048	0.002426	0.00164	0.00163
	0.5	0.01	0.17821	0.017824	0.001816	0.000393	0.00035
		0.03	0.178924	0.017923	0.002077	0.00107	0.001055
0.08	0.9	0.01	0.171327	0.01714	0.001789	0.000543	0.000515
		0.03	0.172681	0.017338	0.002327	0.00157	0.00156
	0.5	0.01	0.17552	0.017555	0.001789	0.000388	0.000346
		0.03	0.17622	0.017652	0.002047	0.001056	0.001041

Other parameter of the AR(1) model: $\delta = 0.06$

assumption (recall the death probability under two assumptions shown in Figure 6.5). In contrast, the endowment insurance is more sensitive to the mortality dependency. As we can see from Figure 6.11, the ratio (dashed line) decreases quickly to below 1 implying that $SD\left[{}_rL^P/m\right]$ under the dependent mortality assumption is smaller than $SD\left[{}_rL^P/m\right]$ under the independent mortality assumption. For the endowment insurance, the risk related to the pure endowment payment dominates the total mortality risk with large portfolio size, (e.g. under our copula mortality assumption, the probability that both (60) and (50) survive 5 years is 0.948, which is the probability of paying the pure endowment benefit for a 5-year endowment insurance). The probability of paying the pure endowment benefit increases as the probability of paying the death benefits decreases. As shown in Figure 6.5, the probability of paying the death benefit under the copula dependent structure is larger than under the independent structure, therefore, the dependent mortality risk of paying the pure endowment benefit is larger than independent mortality risk. Overall, we can see that the effect introducing a dependence term in survival probability is the same for all portfolio size.

To assess the effect of contract initiation ages, Figure 6.12 presents three-dimension plots of the ratio of the limiting volatility $SD\left[{}_rL^P/m\right]$ ($\lim_{m\to\infty} SD\left[{}_rL^P/m\right]$) by male ($x$) ages and female ($y$) ages, where both ($x$) and ($y$) change from 50 to 80. We notice that the pattern

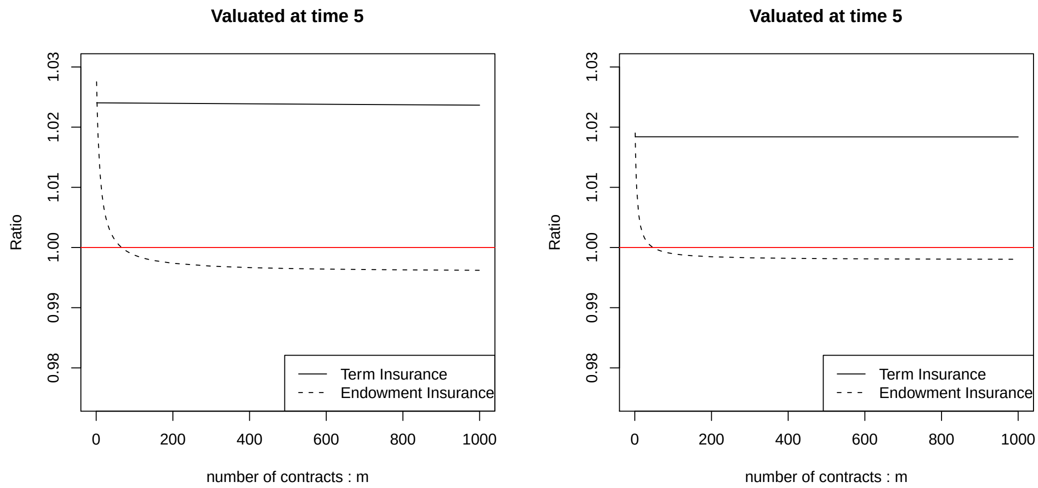


Figure 6.11: Ratio of dependent to independent $SD\left[{}_rL^P/m\right]$ for 10-year term/endowment life insurance contracts valued at year 1/5 under copula assumption

for the ratio of the limiting volatility is similar to the pattern of the expectations under a single policy. To gain an understanding of this similarity, we go back to the decomposition of the total riskiness of the portfolio. In the limiting case, where the size of portfolio goes to infinity, the mortality risks are diversified, the insurance risk dominates the total riskiness. Insurance risk, given by (4.26) or (4.29), can be described as the variability of discounted values weighted by the cash flows, which is similar to the expectation of loss random variable which can be treated as the expectation of the discounted values weighted by the cash flows.

Figure 6.12 also indicates that the limiting volatilities calculated under the copula dependent assumption are smaller than those calculated under the independent assumption for endowment insurance. This suggests that the standard deviation of the loss random variable under the independent assumption is larger than that under the copula mortality assumption for endowment insurance. Therefore, an insurance company would overestimate the standard deviation of the loss random variable with independent mortality assumption for endowment insurance when the existence of dependency of lives can be proven in the portfolio. The case for term insurance is more complicated: the ratios are smaller than one for younger pairs while larger than one for elder pairs. Therefore if the company value the term insurance for younger pairs and elder pairs together in a certain way, the effect of the dependent mortality assumption would be offset.

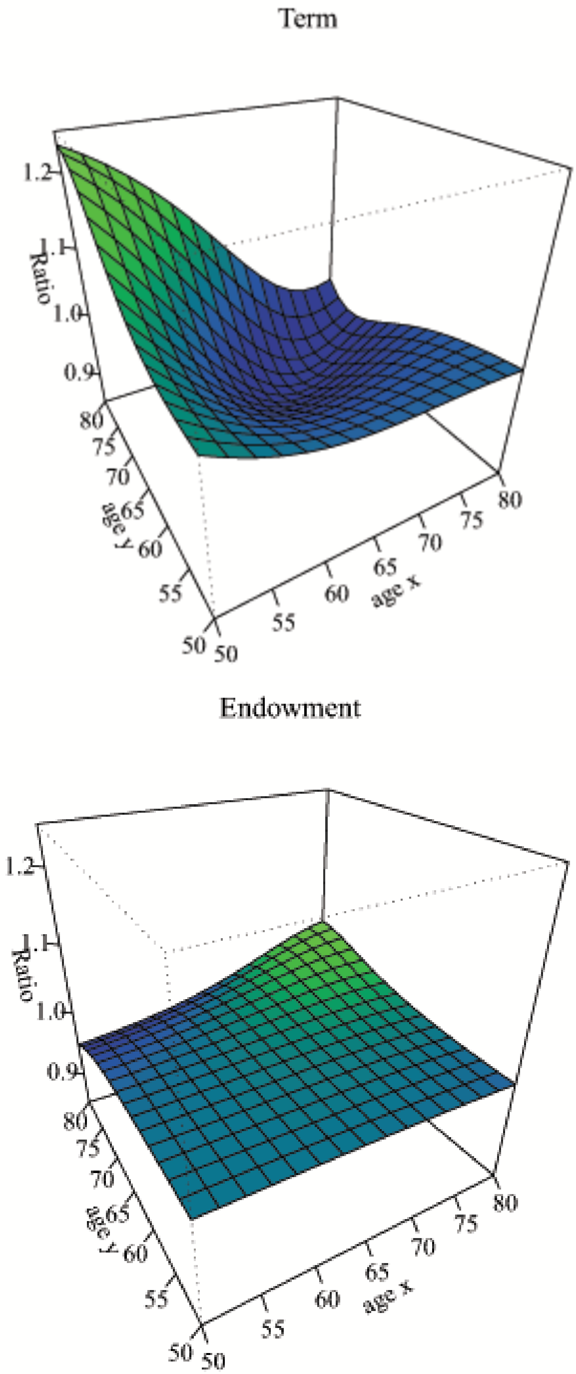


Figure 6.12: Ratio of dependent to independent $\lim_{m\to\infty} SD\left[{}_rL^P/m\right]$ 10-year term/endowment life insurance contracts valued at time 5

Chapter 7

Conclusion

This project explored the valuation of a general (non-homogeneous) joint life insurance portfolio in a stochastic interest rate environment under dependent mortality assumptions. The effect of mortality assumptions on the riskiness of a joint life portfolio is investigated. Two types of joint insurance products, namely joint first-to-die and joint last-to-die, are considered.

Following Parker (1997) and Marceau and Gaillardetz (1999), two methods are used to investigate the behavior of the loss random variable for a joint life insurance portfolio. The first one is based on the individual loss random variables while the second one studies annual stochastic cash flows. General formulas for the first two moments of the prospective loss random variable are derived for a non-homogenous portfolio of joint first-to-die contracts and last-to-die contracts. These formulas can be used with different interest rate models and mortality structures.

The impacts of both the mortality and the interest rate assumptions on the expectations and variances of the prospective loss variables are analyzed for both the single policy and the portfolio. An AR(1) process for interest rate is assumed for our illustration. Three mortality models are considered in this project; the independent model offers simplicity, the common shock model is computationally convenient while Frank's copula model captures fairly well the dependency of lives. In the single policy case, standard deviations of the prospective loss random variable are relatively sensitive to the choice of parameters of the AR(1) process compared to the corresponding expectations; among all four parameters in the rate of return model, the effect of ϕ is the most significant one except the standard deviation of endowment life insurance. The effects of the mortality dependence are assessed.

For the specific data set from Frees et al. (1996), the behavior of loss random variables are similar under common shock and independent mortality assumption. However, the difference between the behavior of loss random variables under Frank's copula and independent mortality assumption can be obvious seen. The loss random variables are also valued at different future times from the contract initiation and the total variability of the loss random variables is decomposed into its insurance risk and investment risk components at each of the valuation dates. This framework provides us with more information about the insurance company's position through the duration of the contracts.

Valuation of a portfolio of contracts is also discussed in this project. The volatility per policy in the portfolio decreases quickly as the portfolio size grows which is due to the diversification of insurance risk (mortality risk). The speed of convergence of the volatility per policy to its limiting value is not sensitive to the choice of mortality models. However, as the portfolio size goes to infinity, the standard deviation per policy in a portfolio is overestimated when using the independent mortality assumption for endowment insurance. In the case of term insurance, the effect of the dependent mortality assumption would be offset when the portfolio is mixed with younger pairs and older pairs a certainty way.

In the course of this project, there are some limitations. Firstly, the parameters for the joint distributions of future lifetimes under independent, Frank's copula and common shock models are taken from Frees et al. (1996). The parameters are estimated from a particular data set, therefore, the results in this project might be sensitive to the underlying data set. Secondly, the parameters for the AR(1) interest rate process are arbitrarily in our project. In practice, they could be estimated from industry data. Thirdly, we ignored expenses and lapses. The model can be more realistic by considering all these factors.

This work can be further continued and extended in many ways. Only the numerical results for a homogenous portfolio are illustrated in our project. It would be interesting to examine a non-homogeneous portfolio. The first two moments of the loss random variable are considered, however, the distribution could also be studied. Besides the behavior of a joint life insurance reserve, the behavior of surplus would be investigated. Finally, the sensitivity of the loss random variable to the dependency parameter under Frank's copula mortality model can be assessed.

Bibliography

[1] J.A. Beekman and C.P Fuelling. Interest and mortality randomness in some annuities. *Insurance: Mathematics and Economics*, 10:275–287, 1991.

[2] F. Black, E. Derman, and W. Toy. A one-factor model of interest rates and its application to treasury bond options. *Financial Analysts Journal*, 46:33–39, 1990.

[3] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, and C.J. Nesbitt. *Actuarial Mathematics*. 2nd. ed. Schaumburg, Illinois : The Society of Actuaries, 1997.

[4] P.P. Boyle. Rates of return as random variables. *Journal of Risk and Insurance*, 43:693–713, 1976.

[5] M.G. Bruno, E. Camerini, and A. Tomassetti. Financial and demographic risks of a portfolio of life insurance policies with stochastic interest rates: Evaluation methods and applications. *North American Actuarial Journal*, 4:44–55, 2000.

[6] A.J.G. Cairns. *Interest Rate Models: An Introduction*. Princeton University Press, 2004.

[7] J. Dhaene. Stochastic interest rates and autoregressive integrated moving average processes. *ASTIN Bulletin*, 19:131–138, 1989.

[8] D.O. Forfar. *Encyclopedia of Actuarial Science*. John Wiley & Sons, New York, 2006.

[9] M.J. Frank. On the simultaneous associativity of $f(x, y)$ and $x+y-f(x, y)$. *Aequationes Mathematicae*, 19:194–226, 1979.

[10] E.W. Frees. Stochastic life contingencies with solvency considerations. *Transactions of Society of Actuaries*, 42:91–148, 1990.

[11] E.W. Frees, J. Carriere, and E. Valdez. Annuity valuation with dependent mortality. *Journal of Risk and Insurance*, 63:229–261, 1996.

[12] P. Gaillardetz. Valuation of life insurance products under stochastic interest rates. *Insurance: Mathematics and Economics*, 42:212–226, 2007.

[13] C. Genest. Frank's family of bivariate distributions. *Biometrika*, 74:549–555, 1987.

[14] C. Genest and J. McKay. The joy of copulas: bivariate distributions with uniform marginals. *American Statistician*, 40:280–283, 1986.

[15] S.L. Lai and E.W. Frees. Examining changes in reserves using stochastic interest models. *Journal of Risk and Insurance*, 62:535–574, 1995.

[16] D. Li. On default correlation: A copula function approach. *Journal of Fixed Income*, 11:245–256, 2000.

[17] X.S. Lin. *Introductory Stochastic Analysis for Finance and Insurance*. John Wiley & Sons, New York, 2006.

[18] N. Lysenko. Stochastic analysis of life insurance surplus. *Master Project, Simon Fraser University*, 2006.

[19] E. Marceau and P. Gaillardetz. On life insurance reserves in a stochastic mortality and interest rates environment. *Insurance: Mathematics and Economics*, 25:261–280, 1999.

[20] R.B. Nelsen. Properties of a one-parameter family of bivariate distributions with specified marginals. *Communications in statistics-Theory and methods*, 15, 1986.

[21] R.B. Nelsen. *An Introduction to Copulas*. Springer, New York, 2006.

[22] R. Norberg. A time-continuous markov chain interest model with applications to insurance. *Journal of Applied Stochastic Models and Data Analysis*, 11:245–256, 1995.

[23] S.M. Pandit and S.M. Wu. *Time Series and System Analysis with Applications*. John Wiley & Sons, New York, 1983.

[24] H.H. Panjer and D.R. Bellhouse. Stochastic modelling of interest rates with applications to life contingencies. *Journal of Risk and Insurance*, 48:91–110, 1980.

[25] H.H. Panjer, P.P. Boyle, S. H. Cox, H.U. Gerber, H.H. Mueller, H.W. Pedersen, S.R. Pliska, M. Sherris, E.S.W. Shiu, and Tan K.S. *Financial Economics with Applications to Investment, Insurance and Pensions*. Society of Actuaries, 1998.

[26] G. Parker. Two stochastic approaches for discounting actuarial functions. *ASTIN Bulletin*, 24:167–181, 1994.

[27] G. Parker. Stochastic analysis of the interaction between investment and insurance risks. *North American Actuarial Journal*, 1:55–84, 1997.

[28] G. Parker. Stochastic interest rates with actuarial applications. *Journal of Applied Stochastic Models and Data Analysis*, 14:335–341, 1998.

[29] A. Sklar. Fonctions de repartitions a n dimensions et leurs marges. *Inst. Statist. Univ., Paris Publ.*, 8:229–231, 1959.

[30] H.R. Waters. The moments and distribution of actuarial functions. *Journal of the Institute of Actuariaes*, 105:Part I: 61–75, 1978.