THE DISCOUNTED PENALTY FUNCTION AND THE DISTRIBUTION OF THE TOTAL DIVIDEND PAYMENTS IN A MULTI-THRESHOLD MARKOVIAN RISK MODEL

by

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Abstract

In this thesis, we study the expected discounted penalty function and the total dividend payments in a risk model with a multi-threshold dividend strategy, where the claim arrivals are modeled by a Markovian arrival process (MAP) and the claim amounts are correlated with the inter-claim times. Systems of integro-differential equations in matrix forms are derived for the expected discounted penalty function and the moments of the discounted dividend payments prior to ruin. A recursive approach based on the integro-differential equations is then provided to obtain the analytical solutions. In addition to the differential approach, by employing some new obtained results in the actuarial literature, another recursive approach with respect to the number of layers is also developed for the expected discounted dividend payments. Examples with exponentially distributed claim amounts are illustrated numerically.

Keywords: Expected discounted penalty function; Discounted dividend payments; Integro-differential equation; Markovian arrival process (MAP); Multi-threshold

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Chapter 1

Introduction

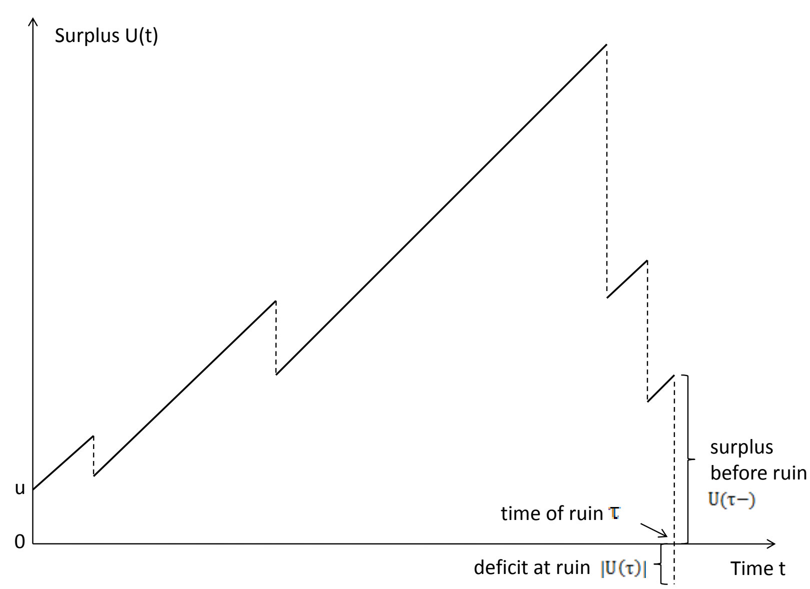
A continuous-time risk model in ruin theory is used to model an insurer's surplus process, $\{U(t); t \geq 0\}$ with an initial surplus $U(0) = u \geq 0$. The process can be modeled as a result of two opposing cash flows, an incoming cash flow of premiums collected continuously at a constant rate $c > 0$ per unit time and an outgoing cash flow of claim amounts $X_1, X_2, \cdots$. This sequence of non-negative claim amount random variables, $\{X_n; n \in \mathbb{N}^+\}$, is assumed to be independent and identically distributed with mean $\mu < \infty$, cumulative distribution function F, probability density function f and its Laplace transform $\hat{f}(s) = \int_0^\infty e^{-sx} f(x)dx$. Let $N(t)$ be the total number of claims up to time t and $S(t) = \sum_{n=1}^{N(t)} X_n$ be the aggregate claims up to time t. Then the surplus process $\{U(t); t \geq 0\}$, illustrated in Figure 1.1, is given by

$$dU(t) = cdt - dS(t), \quad t \geq 0. \tag{1.1}$$

The foundation of the risk model in ruin theory is the so-called classical compound Poisson risk model, in which $\{N(t); t \geq 0\}$ is a Poisson process. The compound Poisson risk model has been studied extensively in actuarial literature. See Gerber (1979), Bower et al. (1997), Asmussen (2000) and reference therein. In recent years, there have been considerable interests in extending results from the classical compound Poisson risk model to other models with more flexible settings. One such extension is called Sparre Andersen risk model, in which the Poisson claim number process is replaced by a more general renewal process. Another extension, which was proposed in Asmussen (1989), is that both the frequency of the claim arrivals and the distribution of the claim amounts are influenced by an external Markovian environment process. It is known as Markov-modulated risk model or the Markovian regime-switching risk model in actuarial literature. Further extension of risk models is to use a Markovian arrival process (MAP; see Neuts (1981)) to govern the external environment process, which has been briefly surveyed in Asmussen (2000). The MAP risk model allows for more complicated non-renewal scenarios with extensive flexibility and includes the Sparre Andersen risk model and the Markov-modulated risk model as special cases.

Allowing dividend payments is another extension for insurance risk models to reflect the surplus cash flows in a more realistic manner. There are three dividend strategies that are of particular interest. The first one is called a constant barrier strategy under which no dividend is paid when the surplus is below a constant barrier but the premiums collected above the barrier are paid out as dividends. The second one is called a threshold dividend strategy under which no dividend is paid when the surplus is below a constant barrier and dividends are paid at a constant rate $d < c$ when the surplus is above the barrier. The third dividend strategy, which is a generalization of the threshold strategy, is called a multi-threshold dividend strategy, under which dividends are paid in a rate that is a step function of the insurer's surplus and most likely an increasing step function. The first two dividend strategies can be considered as special cases of the multi-threshold dividend strategy. Figure A.1, A.2 and A.3 illustrate three sample paths of the surplus process under different dividend strategies. Other dividend strategies include the linear barrier strategy, nonlinear barrier strategy and continuous barrier strategy, with their application in certain areas. In some actuarial literature, dividends are no more payments to shareholders but rather premium discounts that are functions of the surplus level. In this case, dividend strategy is also called surplus-dependent premium policy.

Figure 1.1: Sample path of the surplus process $U(t)$



Introduced by Gerber and Shiu (1998), the Gerber-Shiu discounted penalty function has become an important and standard technical tool in actuarial literature to analyze various ruin-related quantities. Define

$$\tau = \inf\{t \geq 0 : U(t) < 0\} \tag{1.2}$$

to be the first time that the surplus process drops below level 0. It is also called the time of ruin. Let $\delta > 0$ and $w(x, y)$ be a nonnegative function of $x > 0$ and $y > 0$. Here δ can be interpreted as a force of interest or, in the context of Laplace transforms, as a dummy variable. See Gerber and Shiu (1998). Define

$$\phi(u) = \mathbb{E}[e^{-\delta\tau} w(U(\tau-), |U(\tau)|) I(\tau < \infty) | U(0) = u], \quad u \geq 0, \tag{1.3}$$

to be the expected discounted penalty (Gerber-Shiu) function at ruin, given the initial surplus u, where $U(\tau-)$ is the surplus before ruin, $|U(\tau)|$ is the deficit at ruin and $I(\cdot)$ is the indicator function. There are a great variety of ruin-related quantities that can be unified to the expected discounted penalty function, such as the probability of ultimate ruin, the joint distribution of the surplus immediately before ruin and the deficit at ruin, and trivariate Laplace transform of the time of ruin, the surplus immediately before ruin and the deficit at ruin.

In this thesis, we consider the expected discounted penalty function and the distribution of the total dividend payments prior to ruin for a multi-threshold MAP risk model. One typical approach for exploring the expected discounted penalty function in various risk models is the so-called differential approach. Integro-differential equations can be derived and solved analytically for some families of the claim amounts distributions with the help of Laplace transformation. For example, under the multi-threshold dividend strategy, a system of integro-differential equations were derived for the classical compound Poisson risk model in Lin and Sendova (2008) and for the Sparre Andersen risk model in Lu and Li (2009b). Albrecher and Hartinger (2007) argued that the differential approach is rather infeasible when there are large number of layers or more quantities are involved under study from a computational point of view. Alternative recursive approach was proposed for the classical compound Poisson risk model with respect to the number of layers to increase computational feasibility.

The distribution of the total dividend payments prior to ruin and its related quantities are not special cases of the expected discounted penalty function. However, most techniques applied to the problems of dividend payments are basically parallel to those employed in the analysis of the expected discounted penalty function. For example, the distribution of dividend payments prior to ruin for the Sparre Andersen model with generalized Erlang(n) inter-claim times was discussed in Albrecher et al. (2005) and the moments of the dividend payments prior to ruin for the Markov-modulated risk model was discussed in Li and Lu (2007).

The main purpose of this thesis is to show that the differential approach is also applicable for the multi-threshold MAP risk model. By using the matrix-analytic method, systems of integro-differential equations for the expected discounted penalty function and discounted dividend payments are derived and solved analytically and recursively. The layer-based recursive approach, which was introduced in Albrecher and Hartinger (2007) for the classical compound Poisson risk model, is also discussed for the expected discounted dividend payments prior to ruin to increase computational feasibility.

The rest of this thesis is organized as follows. We first review the recent research development of various risk models in Chapter 2 and the main results of the expected discounted penalty function for the classical MAP risk model in Chapter 3. A System of integro-differential equations for the expected discounted penalty function under the multi-threshold dividend strategy is presented in Chapter 4 and a recursive calculation algorithm is provided. The expected value and the moment generating function of the discounted dividend payments prior to ruin under a multi-threshold strategy are considered in Chapter 5. In Chapter 6, an alternative layer-based recursive approach is employed to the expected discounted dividend payments. Finally, numerical examples for two-state models are illustrated in Chapter 7 for the ruin probability and the expected discounted dividend payments prior to ruin when the claim amounts are exponentially distributed.

Chapter 2

Review of Literature

In this chapter, we review some recent research developments on various risk models mentioned in Chapter 1.

2.1 Various Risk Models

2.1.1 Classical Compound Poisson Risk Model

The classical compound Poisson risk model is the foundation of risk model in ruin theory. Recall the surplus process in (1.1) in Chapter 1. The claim counting process $\{N(t); t \geq 0\}$ in the classical compound Poisson risk model is modeled by a time homogeneous Poisson process with parameter λ. It means that the number of claims up to time t, is Poisson distributed with mean λt. For $n = 1, 2, \cdots$, let W_n be the time when the nth claim occurs. Let $Z_1 = W_1$ and $Z_n = W_n - W_{n-1}$ for $n \geq 2$ be the inter-claim times. An important property of the Poisson process implies that $\{Z_n; n \geq 1\}$ is independent and identically exponentially distributed with mean $1/\lambda$. Also it can be proved for aggregate claims $S(t)$ that

$$\mathbb{E}[S(t)] = \mathbb{E}[N(t)]\mathbb{E}[X_n] = (\lambda t)\mu, \quad t \geq 0.$$

Further in the classical compound Poisson risk model, we assume that the premiums received up to time t have a positive loading, that is, $ct > \mathbb{E}[S(t)]$, which implies that $c > \lambda\mu$.

In Lin et al. (2003), the classical compound Poisson risk model under the constant barrier strategy was studied thoroughly. Results regarding the time of ruin and related quantities were derived. Lin and Pavlova (2006) provided a generalization to the classical compound Poisson risk model with a threshold dividend strategy. Two integro-differential equations for the Gerber-Shiu discounted penalty function were derived and solved. Lin and Sendova (2008) further considered a multi-threshold compound Poisson risk model. A piecewise integro-differential equation was derived and a recursive approach for obtaining its solutions was provided. Albrecher and Hartinger (2007) also considered the classical compound Poisson risk model with multi-threshold dividend strategy. A recursive algorithm through the integro-differential equation for the Gerber-Shiu discounted penalty function and the expected discounted dividend payments were derived. An alternative recursive approach with respect to the number of layers was developed to improve the computational feasibility when the number of layers is large.

2.1.2 Sparre Andersen Risk Model

Instead of assuming exponentially distributed independent inter-claim times in the classical compound Poisson risk model, the Sparre Andersen risk model introduced a general distribution function, G, to model the inter-claim times, $\{Z_n; n \geq 1\}$, but retain the assumption of independence.

The Sparre Andersen risk model with Erlang(n) inter-claim times has been studied by Gerber and Shiu (2005). In Albrecher et al. (2005), some results on the distribution of dividend payments for the Sparre Andersen risk model with Erlang(n) inter-claim times under a constant barrier strategy were presented. An integro-differential equation for the moment generating function of the discounted dividend payments was derived. Yang and Zhang (2008) studied the expected discounted penalty function in multi-threshold Sparre Andersen risk model with Erlang(n) inter-claim times.

As discussed in Li (2008), it is common in actuarial literature that the phase-type distribution is considered as a distribution for the inter-claims. The phase-type distribution is dense in the field of all positive-valued distributions, so that it can be used to approximate any positive valued distribution. It includes combinations and mixtures of exponential and Erlang distributions as special cases. Consider a terminating Markov process $\{J(t); t \geq 0\}$ with $m+1$ states such that states $1, \cdots, m$ are transient states and state $m+1$ is an absorbing state. The phase-type distribution $PH(\vec{a}, \boldsymbol{S})$ is the distribution of time of the process starting to the absorbing state, in which $\vec{a} = (a_1, \cdots, a_m)^\top$ is the initial entrance probability vector with $\sum_{i=1}^m a_i = 1$ and $\boldsymbol{S} = (s_{i,j})_{i,j=1}^m$ is the restriction to a state space $E = \{1, 2, \cdots, m\}$ of the transition rate matrix

$$\begin{pmatrix} \boldsymbol{S} & \vec{s} \\ 0 & 0 \end{pmatrix}$$

on $E \cup \{0\}$ with 0 as the absorbing state and $\vec{s} = -\boldsymbol{S}\vec{1}$ as a vector of exit rates, where $\vec{1}$ is the column vector of ones. More precisely, we have $s_{i,i} < 0$, $s_{i,j} \geq 0$ for $i \neq j$, and $\sum_{j=1}^m s_{i,j} \leq 0$ for any $i \in E$. Indeed the distribution function of inter-claim times, $\{Z_n; n \geq 1\}$, is given by,

$$F(t) = 1 - \vec{a}^\top e^{t\boldsymbol{S}}\vec{1}, \quad t \geq 0.$$

The density function is

$$f(t) = \vec{a}^\top e^{t\boldsymbol{S}}\vec{s}, \quad t \geq 0,$$

and the Laplace transform with parameter r is

$$\hat{f}(r) = \vec{a}^\top (r\boldsymbol{I} - \boldsymbol{S})^{-1}\vec{s},$$

where $\boldsymbol{I}$ is the identity matrix. A detailed introduction to phase-type distribution, associated properties and applications in ruin theory can be found in Neuts (1981), Latouche and Ramaswami (1999), Asmussen (2000) and references therein. Similar to the positive safety loading condition in the classical compound Poisson risk model, we assume that $c\mathbb{E}[Z] > \mathbb{E}[X]$.

By using both renewal theory and Markovian techniques, results in the classical compound Poisson risk model can be extended to the Sparre Andersen risk model with phase-type inter-claim times in compact matrix forms. For example, Li (2008b) considered the problem that the surplus hits a certain level and the time of ruin in a Sparre Andersen risk model with phase-type inter-claim times. The same problem was studied by Gerber (1990) for the classical compound Poisson risk model. A matrix expression for the expected discounted dividend payments was given in the presence of a constant dividend strategy. Lu and Li (2009b) used the matrix-analytic method to extend the results in Lin and Sendova (2008) to a multi-threshold Sparre Andersen risk model with phase-type inter-claim times. The matrix-form piecewise integro-differential equation was derived and the analytical solution to this equation was obtained.

2.1.3 Markov-modulated Risk Model

Asmussen (1989) introduced a risk model that both the frequency of the claim arrivals and the distribution of the claim amounts are not homogeneous in time but determined by an external Markov process $\{J(t); t \geq 0\}$. As pointed out in Li and Lu (2008), the motivation for this generalization is to enhance the flexibility in modeling the claim arrivals and the distribution of the claim amounts in the classical risk model. Examples of how such a mechanism could be relevant in ruin theory are usually given as weather conditions (e.g., normal and icy road conditions) in automobile insurance portfolios and epidemic outbreaks in health insurance portfolios. Zhu and Yang (2008) referred to states of the external process as economic circumstances or political regime switchings.

In Asmussen (1989), it was supposed that $\{J(t); t \geq 0\}$ is a homogeneous, irreducible and recurrent Markov process with a finite state space $E = \{1, \cdots, m\}$. The intensity matrix governing $\{J(t); t \geq 0\}$ is denoted by $\boldsymbol{\Lambda} = (\alpha_{i,j})_{i,j=1}^m$, where $\alpha_{i,i} = \sum_{i \neq j}^m \alpha_{i,j}$ for $i \in E$. Its stationary limiting distribution, denoted by $\vec{\pi} = (\pi_1, \cdots, \pi_m)^\top$, can be computed as the positive solution of $\vec{\pi}^\top \mathbf{\Lambda} = \vec{0}^\top$ and $\vec{\pi}^\top \vec{1} = 1$. When $J(t) = i$, the number of claims occurring in an infinitesimal time interval $(t, t+h]$, $N(t+h) - N(t)$ is assumed to follow a Poisson distribution with parameter $\lambda_i h$ and the corresponding claim amounts are distributed with mean μ_i, cumulative distribution function F_i and probability density function f_i for $i \in E$. We also assume that the positive loading condition holds, $\sum_{i=1}^m \pi_i(c - \lambda_i \mu_i) > 0$.

In the Markov-modulated risk model, a system of integro-differential equations satisfied by the nth moment of the discounted dividend payments was derived and solved under a constant dividend strategy in Li and Lu (2007). The expected discounted penalty functions and their decompositions in the same risk model were investigated in Li and Lu (2008). Lu and Li (2009a) presented the matrix form of systems of integro-differential equations for the expected discounted penalty function and the moments of the total dividend payments in a Markov-modulated risk model with a threshold dividend strategy.

2.2 MAP Risk Model

The so-called MAP risk model is the one where the environment (external) process $\{J(t); t \geq 0\}$ is assumed to be a MAP with representation $(\vec{\alpha}, \boldsymbol{D_0}, \boldsymbol{D_1})$, which has recently been investigated by some researchers. See, for example, Badescu (2008), Ren (2009), and Cheung and Landriault (2009). The initial distribution is $\vec{\alpha} = (\alpha_1, \cdots, \alpha_m)^\top$, and the intensity matrix is $\boldsymbol{D_0} + \boldsymbol{D_1}$. The intensity of a state changing from state i to state $j \neq i$ in $E = \{1, \cdots, m\}$ without an accompanying claim is given by the (i,j)th element of $\boldsymbol{D_0}$, namely, $D_0(i,j) \geq 0$. The intensity of a state changing from i to state j (possibly $j = i$) in E with an accompanying claim is given by $D_1(i,j) \geq 0$. The diagonal elements of $\boldsymbol{D_0}$ are negative values such that the sum of the elements on each row of the matrix $\boldsymbol{D_0} + \boldsymbol{D_1}$ are all zeros. The sequence of claim amounts $\{X_n; n \in \mathbb{N}^+\}$ is assumed to be distributed with cumulative distribution function $F_{i,j}$, probability density function $f_{i,j}$ and its Laplace transform $\hat{f}_{i,j}(s)$, where i is the state when a claim occurs and j is the state immediately after a claim occurs.

The MAP risk model contains the classical compound Poisson risk model, Markov-modulated risk model and Sparre Andersen risk model with phase-type inter-claim times above as special cases. When $m = 1$, $\boldsymbol{D_0} = -\lambda$ and $\boldsymbol{D_1} = \lambda$, the MAP risk model reduces to the classical compound Poisson risk model. When $\boldsymbol{D_0} = \boldsymbol{S}$ and $\boldsymbol{D_1} = -\boldsymbol{S}\vec{1}\vec{a}^\top$, the MAP risk model simplifies to the Sparre Andersen risk model with $PH(\vec{a}, \boldsymbol{S})$ as the distribution for inter-claim times. When $\boldsymbol{D_1} = \text{diag}[\lambda_1, \cdots, \lambda_m]$ and $\boldsymbol{D_0} = \mathbf{\Lambda} - \boldsymbol{D_1}$, the MAP risk model becomes to the Markov-modulated risk model.

Examples of the MAP risk model related to practical issues were discussed in Badescu et al. (2005). For instance, the "contagion" example assumes that claim behavior is influenced by two environmental states. Environment A corresponds to a “normal”situation with standard claim rates and claim sizes, while environment B reflects periods of contagion, when a highly infectious disease is causing a supplemental stream of claims. Also the non-renewal generalization of the Sparre Andersen risk model is an example of the MAP risk model. Instead of assuming that the state is sampled anew at the end of each payment, we now need to keep tack of the MAP state during each payment interval.

As mentioned in Asmussen (2000), the MAP has received much attention in the queuing literature, though this still remains to be implemented in ruin theory. Ahn and Badescu (2007) analyzed the expected discounted penalty function in the MAP risk model and derived a defective renewal equation in a matrix form. The use of the busy period distribution for the canonical fluid flow model is a key factor in their paper. Badescu et al. (2005) analyzed a multi-threshold MAP risk model, linking an insurer’s surplus process to an embedded fluid flow process. In general, the fluid flow type analysis of the MAP risk models relies heavily on the knowledge of the Laplace transform of first passage times and the infinitesimal generator. The role played by the roots of Lundberg’s generalized equation in the differential approach is replaced by the determination of the Laplace transform of a particular busy period arising in an unbounded fluid queue. A drawback of this fluid flow analyses in the ruin theory application is its limitation to the claim amount distributions that are phase-type distributions, resulting in poor performance of fitting for heavy-tailed distributions in general.

Compared to those results, Cheung and Landriault (2009) proposed to rely on a purely analytic approach to analyze MAP risk model. Moments of the discounted dividend payments for a MAP risk model under a constant barrier strategy were studied. Also, the MAP risk model with a dynamic barrier level that depends on the state of the underlying environment was considered. Badescu (2008) derived the integro-differential equation and solved the initial value at $u = 0$ for the MAP risk model with no dividend strategy involved. Ren (2009) further derived a matrix expression for the Laplace transform of the first time that the surplus process reaches a given target from the initial surplus.

Chapter 3

Preliminary

In this chapter, we first introduce the notation used in the thesis for the multi-threshold MAP risk model. Then the main results related to the MAP risk model with no dividend strategy involved are presented. As we will see in the next chapter, the expected discounted penalty function for the multi-threshold MAP risk model is associated with the expected discounted penalty function for a MAP risk model with no dividend strategy involved.

In the MAP risk model discussed in this thesis, it is assumed that there are n thresholds $0 < b_1 < \cdots < b_n < \infty$ with $b_0 = 0$ and $b_{n+1} = \infty$, such that when the surplus is between the thresholds b_{k-1} and b_k, the dividend rate is d_k and the corresponding premium rate is $c_k = c - d_k$, for $k = 1, \cdots, n+1$, where $c = c_1 > c_2 > \cdots > c_n > c_{n+1} \geq 0$ is assumed. To separate this special case from the general case without dividend strategy, we let $B = \{b_1, \cdots, b_n\}$ be the multi-threshold setting, τ_B be the time of ruin and $\{U_B(t); t \geq 0\}$ be the surplus process under the multi-threshold model with initial surplus $U_B(0) = u$. Then similar to (1.1), $\{U_B(t); t \geq 0\}$ satisfies the following stochastic differential equation, for $k = 1, \cdots, n+1$,

$$dU_B(t) = c_k dt - dS(t), \quad b_{k-1} \leq U_B(t) < b_k. \tag{3.1}$$

For notation convenience, let $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | J(0) = i]$. Then define

$$\phi_i(u; B) = \mathbb{E}_i[e^{-\delta \tau_B} w(U_B(\tau_B-), |U_B(\tau_B)|) I(\tau_B < \infty) | U_B(0) = u], \quad u \geq 0, i \in E, \tag{3.2}$$

to be the expected discounted penalty function in the multi-threshold MAP risk model, given the initial surplus u and the initial MAP state $i \in E$. Then

$$\phi(u; B) = \vec{\alpha}^\top \vec{\phi}(u; B), \quad u \geq 0,$$

where $\vec{\phi}(u; B) = (\phi_1(u; B), \cdots, \phi_m(u; B))^\top$. When $\delta = 0$ and $w(x, y) = 1$, $\phi(u; B)$ and $\phi_i(u; B)$ are simplified to the ruin probabilities in the MAP risk model, given the initial surplus u and the initial MAP phase i, which are denoted as $\varphi(u; B)$ and $\varphi_i(u; B)$, respectively.

When $b_1 = \infty$, the risk model given by (3.1) reduces to the one in (1.1) with no dividend strategy involved. The expected discounted penalty function in the MAP risk model with no dividend strategy involved is defined as

$$\phi_i(u) = \mathbb{E}_i[e^{-\delta\tau} w(U(\tau-), |U(\tau)|) I(\tau < \infty) | U(0) = u], \quad u \geq 0, i \in E, \tag{3.3}$$

given the initial surplus u and the initial MAP state $i \in E$. Similarly, we have $\vec{\phi}(u) = (\phi_1(u), \cdots, \phi_m(u))^\top$.

3.1 Integro-differential Equation for $u \geq 0$

In the section, we first review the results in Badescu (2008) that the expected discounted penalty function given in (3.3) satisfies a system of integro-differential equations. Then we present a lemma which can be applied to find a general solution to this system of integro-differential equations for the case of $u \geq 0$.

By conditioning on the events occurring in an infinitesimal time interval, it was derived in Badescu (2008) that the expected discounted penalty function (3.3) satisfies

$$c\phi_i'(u) = \delta\phi_i(u) - \sum_{j=1}^{m} D_0(i,j)\phi_j(u) - \sum_{j=1}^{m} D_1(i,j) \left(\int_0^u \phi_j(u-x) dF_{i,j}(x) - \omega_{i,j}(u) \right), \quad u \geq 0, \tag{3.4}$$

where

$$\omega_{i,j}(u) = \int_u^\infty w(u, x-u) dF_{i,j}(x). \tag{3.5}$$

In the matrix notation, equation (3.4) can be rewritten as

$$c\vec{\phi}'(u) = \delta\vec{\phi}(u) - \boldsymbol{D_0}\vec{\phi}(u) - \int_0^u \boldsymbol{\Lambda_f}(x)\vec{\phi}(u-x)dx - \vec{\zeta}(u), \quad u \geq 0, \tag{3.6}$$

where $\boldsymbol{\Lambda_f}(x)$ is a matrix with the (i,j)th element given by $D_1(i,j)f_{i,j}(x)$ and $\vec{\zeta}(u)$ is a column vector with the ith element given by $\zeta_i(u) = \sum_{j=1}^m D_1(i,j)\omega_{i,j}(u)$. Note that equation (3.6) is a non-homogeneous integro-differential equation in matrix form and its corresponding homogeneous integro-differential equation in matrix form is

$$c\vec{\phi}'(u) = \delta\vec{\phi}(u) - \boldsymbol{D_0}\vec{\phi}(u) - \int_0^u \boldsymbol{\Lambda_f}(x)\vec{\phi}(u-x)dx, \quad u \geq 0. \tag{3.7}$$

Taking the Laplace transform of both sides of (3.7), we have, on the left hand side,

$$\begin{aligned} \int_0^\infty e^{-su} c\vec{\phi}'(u)du &= c\int_0^\infty e^{-su} d\vec{\phi}(u) \\ &= ce^{-su}\vec{\phi}(u)\Big|_0^\infty + s\int_0^\infty e^{-su}\vec{\phi}(u)du \\ &= -c\vec{\phi}(0) + s\mathcal{L}_s[\vec{\phi}(u)], \end{aligned}$$

and, on the right hand side,

$$\begin{aligned}
&\int_0^\infty e^{-su}\left[\delta\vec{\phi}(u)-\boldsymbol{D_0}\vec{\phi}(u)-\int_0^u \boldsymbol{\Lambda_f}(x)\vec{\phi}(u-x)dx\right]du\\
=\;&(\delta\boldsymbol{I}-\boldsymbol{D_0})\mathcal{L}_s\left[\vec{\phi}(u)\right]-\int_0^\infty\int_x^\infty e^{-su}\boldsymbol{\Lambda_f}(x)\vec{\phi}(u-x)dudx\\
=\;&(\delta\boldsymbol{I}-\boldsymbol{D_0})\mathcal{L}_s\left[\vec{\phi}(u)\right]-\int_0^\infty \boldsymbol{\Lambda_f}(x)\int_0^\infty e^{-s(y+x)}\vec{\phi}(y)dydx\\
=\;&(\delta\boldsymbol{I}-\boldsymbol{D_0})\mathcal{L}_s[\vec{\phi}(u)]-\boldsymbol{\Lambda_{\hat{f}}}(s)\mathcal{L}_s[\vec{\phi}(u)].
\end{aligned}$$

Finally, we obtain,

$$\left[\left(s-\frac{\delta}{c}\right)\boldsymbol{I}+\frac{1}{c}(\boldsymbol{D_0}+\boldsymbol{\Lambda_{\hat{f}}}(s))\right]\mathcal{L}_s\left[\vec{\phi}(u)\right]=\vec{\phi}(0), \tag{3.8}$$

where $\boldsymbol{\Lambda_{\hat{f}}}(s)$ is a matrix with the (i,j)th element given by $D_1(i,j)\hat{f}_{i,j}(s)$ and $\mathcal{L}_s$ is the Laplace transform operator for a column vector of functions with a complex argument s. That is, $\mathcal{L}_s[\vec{\phi}(u)]=\left(\hat{\phi}_1(s),\cdots,\hat{\phi}_m(s)\right)^\top$.

Let

$$\boldsymbol{L}_c(s)=\left(s-\frac{\delta}{c}\right)\boldsymbol{I}+\frac{1}{c}(\boldsymbol{D_0}+\boldsymbol{\Lambda_{\hat{f}}}(s)). \tag{3.9}$$

Assuming that its inverse exists, then equation (3.8) can be rewritten as

$$\mathcal{L}_s[\vec{\phi}(u)]=\left[\left(s-\frac{\delta}{c}\right)\boldsymbol{I}+\frac{1}{c}(\boldsymbol{D_0}+\boldsymbol{\Lambda_{\hat{f}}}(s))\right]^{-1}\vec{\phi}(0). \tag{3.10}$$

Further let

$$\mathbf{v}(u)=\mathcal{L}_s^{-1}\left\{\left[\left(s-\frac{\delta}{c}\right)\boldsymbol{I}+\frac{1}{c}(\boldsymbol{D_0}+\boldsymbol{\Lambda_{\hat{f}}}(s))\right]^{-1}\right\} \tag{3.11}$$

be the Laplace inversion of the inverse of matrix $\boldsymbol{L}_c(s)$. Then $\mathcal{L}_s[\mathbf{v}(u)]$ is the Laplace transform of $\mathbf{v}(u)$. By (3.10) and (3.11), we get a general solution for the homogeneous integro-differential equation (3.7) as,

$$\vec{\phi}(u)=\mathbf{v}(u)\vec{\phi}(0),\quad u\geq 0.$$

Setting $u=0$ in (3.10) and substituting with (3.11), we have

$$\mathcal{L}_s[\vec{\phi}(0)]=\mathcal{L}_s[\mathbf{v}(0)]\vec{\phi}(0).$$

Thus $\mathbf{v}(0)=\boldsymbol{I}$.

Now we present the analytical expression for $\vec{\phi}(u)$ satisfying the non-homogeneous integro-differential equation (3.6) in the following lemma.

Lemma 1 *The solution to* (3.6) *is*

$$\vec{\phi}(u)=\mathbf{v}(u)\vec{\phi}(0)-\frac{1}{c}\int_0^u \mathbf{v}(u-t)\vec{\zeta}(t)dt,\quad u\geq 0 \tag{3.12}$$

where $\mathbf{v}(u)$ *is given by* (3.11) *and the expression for* $\vec{\phi}(0)$ *is to be discussed in Section 3.4.*

Proof. Similar to deriving equation (3.8), taking the Laplace transform of both sides of (3.6) yields

$$\left[\left(s - \frac{\delta}{c}\right)\boldsymbol{I} + \frac{1}{c}(\boldsymbol{D_0} + \boldsymbol{\Lambda}_{\hat{\boldsymbol{f}}}(s))\right]\mathcal{L}_s[\vec{\phi}(u)] = \vec{\phi}(0) - \frac{1}{c}\mathcal{L}_s[\vec{\zeta}(u)]. \tag{3.13}$$

By (3.11), we get

$$\begin{aligned}\mathcal{L}_s[\vec{\phi}(u)] &= \mathcal{L}_s\left[\mathbf{v}(u)\right]\left[\vec{\phi}(0) - \frac{1}{c}\mathcal{L}_s[\vec{\zeta}(u)]\right] \\ &= \mathcal{L}_s\left[\mathbf{v}(u)\vec{\phi}(0)\right] - \frac{1}{c}\mathcal{L}_s\left[\int_0^u \mathbf{v}(u-t)\vec{\zeta}(t)dt\right].\end{aligned}$$

So $\vec{\phi}(u)$ can be evaluated as (3.12). □

Note that this lemma is parallel to Lemma 1 in Lu and Li (2009a) for the Sparre Andersen risk model.

3.2 Integro-differential Equation for $u \geq b_{k-1}$

In this section, we extend the result in Lemma 1 to the case that $u \geq b_{k-1}$, for $k = 1, 2, \cdots, n+1$, where b_{k-1} is the threshold point of the kth layer in the multi-threshold MAP risk model. That is, we consider the integro-differential equation in the following form:

$$c_k\vec{\phi}_k'(u) = \delta\vec{\phi}_k(u) - \boldsymbol{D_0}\vec{\phi}_k(u) - \int_0^{u-b_{k-1}} \boldsymbol{\Lambda}_{\boldsymbol{f}}(t)\vec{\phi}_k(u-t)dt - \vec{\zeta}_k(u), \quad u \geq b_{k-1}. \tag{3.14}$$

Later in Chapter 4, we will see that $\vec{\phi}_k(u)$ in (3.14), with a special setting of $\vec{\zeta}_k(u)$, is associated with the expected discounted penalty function for the multi-threshold MAP risk model $\vec{\phi}_k(u; B)$ when $b_{k-1} \leq u \leq b_k$.

Using the same techniques in the proof of Lemma 1, we present an analytical expression for $\vec{\phi}_k(u)$ in the following lemma.

Lemma 2 *The solution to* (3.14) *is*

$$\vec{\phi}_k(u) = \mathbf{v}_k(u-b_{k-1})\vec{\phi}_k(b_{k-1}) - \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\zeta}_k(u-t)dt, \quad u \geq b_{k-1}, \tag{3.15}$$

where

$$\mathbf{v}_k(u-b_{k-1}) = \mathcal{L}_s^{-1}\left\{\left[\left(s - \frac{\delta}{c_k}\right)\boldsymbol{I} + \frac{1}{c_k}(\boldsymbol{D_0} + \boldsymbol{\Lambda}_{\hat{\boldsymbol{f}}}(s))\right]^{-1}\right\}, \quad u \geq b_{k-1}, \tag{3.16}$$

with $\mathbf{v}_k(0) = \boldsymbol{I}$ *and* $\vec{\phi}_k(b_{k-1})$ *is to be discussed in Section 3.4.*

Proof. When $u \geq b_{k-1}$, letting $x = u - b_{k-1}$, we can rewrite $\vec{\phi}_k(u)$ in (3.14) as

$$c_k\vec{\phi}_k'(x+b_{k-1}) = \delta\vec{\phi}_k(x+b_{k-1}) - \boldsymbol{D_0}\vec{\phi}_k(x+b_{k-1}) \tag{3.17}$$

$$- \int_0^x \boldsymbol{\Lambda_f}(t)\vec{\phi}_k(x+b_{k-1}-t)dt - \vec{\zeta}_k(x+b_{k-1}), \quad x \geq 0. \tag{3.18}$$

Further letting $\vec{\phi}_k^*(x) = \vec{\phi}_k(x+b_{k-1})$ and $\vec{\zeta}_k^*(x) = \vec{\zeta}_k(x+b_{k-1})$, (3.17) can be rewritten as

$$c_k\vec{\phi}_k^{*\prime}(x) = \delta\vec{\phi}_k^*(x) - \boldsymbol{D_0}\vec{\phi}_k^*(x) - \int_0^x \boldsymbol{\Lambda_f}(t)\vec{\phi}_k^*(x-t)dt - \vec{\zeta}_k^*(x), \quad x \geq 0. \tag{3.19}$$

Applying Lemma 1 to the equation above, we obtain that the solution to (3.19) is

$$\begin{aligned} \vec{\phi}_k^*(x) &= \mathbf{v}_k(x)\vec{\phi}_k^*(0) - \frac{1}{c_k}\int_0^x \mathbf{v}_k(x-t)\vec{\zeta}_k^*(t)dt \\ &= \mathbf{v}_k(x)\vec{\phi}_k^*(0) - \frac{1}{c_k}\int_0^x \mathbf{v}_k(t)\vec{\zeta}_k^*(x-t)dt, \quad x \geq 0, \end{aligned} \tag{3.20}$$

where

$$\mathbf{v}_k(x) = \mathcal{L}_s^{-1}\left\{\left[\left(s - \frac{\delta}{c_k}\right)\boldsymbol{I} + \frac{1}{c_k}(\boldsymbol{D_0} + \boldsymbol{\Lambda_{\hat{f}}}(s))\right]^{-1}\right\}, \quad x \geq 0$$

with $\mathbf{v}_k(0) = \boldsymbol{I}$. Then with $x = u - b_{k-1}$, equation (3.20) is

$$\vec{\phi}_k(u) = \mathbf{v}_k(u-b_{k-1})\vec{\phi}_k(b_{k-1}) - \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\zeta}_k(u-t)dt, \quad u \geq b_{k-1},$$

which is the expression of (3.15). □

3.3 Evaluation of $\mathbf{v}_k(u)$

As we can see from the results above, the evaluation of $\mathbf{v}(u)$ and $\mathbf{v}_k(u)$ is crucial in obtaining the explicit results for $\vec{\phi}(u)$ and $\vec{\phi}_k(u)$, respectively. In this section, we begin with the evaluation of $\mathbf{v}(u)$. Then $\mathbf{v}_k(u)$ can be obtained by the same argument with c being replaced by c_k.

One approach of evaluating $\mathbf{v}(u)$ is to find the Laplace inversion directly from (3.11). Usually it is difficult to get the explicit expression for the Laplace inversion of a matrix inversion. In Li and Lu (2007), it was shown that in a two-state example it is possible to obtain the explicit expression of the Laplace inversion of the inverse of a matrix for the Markov-modulated model with the rational family of claim amount distributions.

The other approach is to involve the evaluation of the special expected discounted penalty function. This approach was first discussed in Gerber and Shiu (1998) for the classical compound Poisson risk model, and was developed in Li (2008b) for the Sparre-Andersen risk model with phase-type inter-claim times. Now we show in the following that this approach can also be adopted for the MAP risk model. We first introduce the concept of stopping time $\tau^*(u,a,b)$, for $a \leq b$, which is defined as

$$\tau^*(u,a,b) = \inf\{t \geq 0 : U(t) \notin [a,b] | U(0) = u\}.$$

Define

$$\tau^+(u,a,b) = \begin{cases} \tau^*(u,a,b) & \text{if } U(\tau^*(u,a,b)) = b, \\ \infty & \text{if } U(\tau^*(u,a,b)) < a, \end{cases} \tag{3.21}$$

and

$$\tau^-(u,a,b) = \begin{cases} \infty & \text{if } U(\tau^*(u,a,b)) = b, \\ \tau^*(u,a,b) & \text{if } U(\tau^*(u,a,b)) < a. \end{cases} \tag{3.22}$$

The stopping time $\tau^+(u,a,b)$ can be interpreted as the first time of exiting through upper level b and $\tau^-(u,a,b)$ as the first time of dropping below lower level a.

Let

$$B_{i,j}(u,b) = \mathbb{E}[e^{-\delta\tau^+(u,0,b)} I(J(\tau^+(u,0,b)) = j) | J(0) = i], \quad u \geq 0, \tag{3.23}$$

be the Laplace transform of $\tau^+(u,0,b)$, given that the surplus reaches the level b in MAP phase j, the initial surplus is u and the initial state is i. When $b = 0$, the surplus process drops below level 0 before upcrossing it again. Denoted by $\tau_0 = \tau^+(u,0,0)$, the time of the first upcrossing of the surplus process through level 0 after the time of ruin, is also called the time of recovery. For $\delta > 0$, define

$$\psi_{i,j}(u) = \mathbb{E}_i[e^{-\delta\tau_0} I(\tau < \infty, J(\tau_0) = j) | U(0) = u], \quad u \geq 0,$$

to be the Laplace transform of the time of recovery if the process upcrosses level 0 at state j after ruin given the initial surplus u and the initial state i.

Let $\boldsymbol{B}(u;b)$ be a matrix with the (i,j)th element being $B_{i,j}(u;b)$. It was shown in Ren (2009) that $\boldsymbol{B}(u;b)$ has the form

$$\boldsymbol{B}(u;b) = e^{-\boldsymbol{K}(b-u)}, \quad u \leq b,$$

where matrix $\boldsymbol{K}$ satisfies the following matrix equation:

$$c\boldsymbol{K} + (-\delta\boldsymbol{I} + \boldsymbol{D_0}) + \int_0^\infty \boldsymbol{\Lambda}_{\hat{\boldsymbol{f}}}(x) e^{-\boldsymbol{K}x} dx = 0. \tag{3.24}$$

The solution to (3.24) was further shown in Ren (2009) as

$$\boldsymbol{K} = \boldsymbol{H}\Delta_\rho \boldsymbol{H}^{-1}, \tag{3.25}$$

where $\Delta_\rho = \text{diag}[\rho_1, \cdots, \rho_m]$, with $\rho_1, \cdots, \rho_m$ being the solutions to the equation $\det[\boldsymbol{L}_c(s)] = 0$ in the right half complex plain and $\boldsymbol{H} = (\vec{h}_1, \vec{h}_2, \cdots, \vec{h}_m)$ with column vector $\vec{h}_i$ being an eigenvector of $\boldsymbol{L}_c(\rho_i)$ corresponding to eigenvalue 0, that is $\boldsymbol{L}_c(\rho_i)\vec{h}_i = \vec{0}$, for $i \in E$. For the proof of the existence of those roots, see Gerber and Shiu (1998) for the classical compound Poisson risk model, Li and Garrido (2005) for the Sparre Andersen risk model with Erlang (n) inter-claim times, Li (2008a) for the Sparre Andersen risk model with phase-type inter-claim times, and Lu and Tsai (2007) for the Markov-modulated risk model.

Let $\boldsymbol{\psi}(u)$ be a matrix with the (i,j)th element being $\psi_{i,j}(u)$. It was shown in Theorem 1 in Li (2008b) that

$$\boldsymbol{\psi}(u) = \mathbb{E}\left[e^{-\delta\tau + \boldsymbol{K}U(\tau)} I(\tau < \infty) | U(0) = u\right], \quad u \geq 0.$$

Substituting matrix $\boldsymbol{K}$ in (3.25) into the equation above, we can rewrite $\boldsymbol{\psi}(u)$ as

$$\boldsymbol{\psi}(u) = \boldsymbol{H}\, \mathrm{diag}[\theta_1(u), \cdots, \theta_m(u)] \boldsymbol{H}^{-1}, \tag{3.26}$$

where

$$\theta_i(u) = \mathbb{E}\left[e^{-\delta\tau - \rho_i |U(\tau)|} I(\tau < \infty) | U(0) = u\right], \quad i \in E. \tag{3.27}$$

As pointed out in Lu and Li (2009b), $\theta_i(u)$ is the bivariate Laplace transformation with respect to the time of ruin and the deficit at ruin with parameters δ and ρ_i. It is also a special case of the expected discounted penalty function with $w(x,y) = e^{-\rho_i y}$.

Using the same methodology in Section 6 of Gerber and Shiu (1998) and Section 4 of Li (2008b), $\boldsymbol{B}(u;b)$ has the following expression

$$\boldsymbol{B}(u;b) = [e^{\boldsymbol{K}u} - \boldsymbol{\psi}(u)][e^{\boldsymbol{K}b} - \boldsymbol{\psi}(b)]^{-1}, \quad u \geq 0,$$

and it also satisfies a homogeneous integro-differential equation in matrix form as

$$c\boldsymbol{B}'(u;b) = \delta \boldsymbol{B}(u;b) - \boldsymbol{D_0}\boldsymbol{B}(u;b) - \int_0^u \boldsymbol{\Lambda_f}(x)\boldsymbol{B}(u-x;b)dx, \quad u \geq 0, \tag{3.28}$$

with $\boldsymbol{B}(b;b) = \boldsymbol{I}$.

Now let

$$\mathbf{v}(u) = [e^{\boldsymbol{K}u} - \boldsymbol{\psi}(u)][\boldsymbol{I} - \boldsymbol{\psi}(0)]^{-1}, \quad u \geq 0. \tag{3.29}$$

Substituting $\boldsymbol{B}(u;b)$ with $\mathbf{v}(u)$ in (3.28), we then have

$$c\mathbf{v}'(u) = \delta\mathbf{v}(u) - \boldsymbol{D_0}\mathbf{v}(u) - \int_0^u \boldsymbol{\Lambda_f}(x)\mathbf{v}(u-x)dx, \quad u \geq 0, \tag{3.30}$$

with $\mathbf{v}(0) = \boldsymbol{I}$. Taking the Laplace transform of both sides of (3.30) and after some manipulations, we arrive that $\mathbf{v}(u)$ has the same form as (3.11). So $\mathbf{v}(u)$ can be evaluated with the help of $\psi(u)$ in (3.29). By the expression of $\boldsymbol{K}$ and $\boldsymbol{\psi}(u)$ given in (3.25) and (3.26), respectively, we can further write $\mathbf{v}(u)$ as

$$\begin{aligned} \mathbf{v}(u) &= \boldsymbol{H}\left[e^{\Delta_\rho} - \mathrm{diag}[\theta_1(u), \cdots, \theta_m(u)]\right] \boldsymbol{H}^{-1}\boldsymbol{H}\left[\boldsymbol{I} - \mathrm{diag}[\theta_1(0), \cdots, \theta_m(0)]\right]^{-1} \boldsymbol{H}^{-1} \\ &= \boldsymbol{H}\, \mathrm{diag}\left[\frac{e^{\rho_1 u} - \theta_1(u)}{1-\theta_1(0)}, \cdots, \frac{e^{\rho_m u} - \theta_m(u)}{1-\theta_m(0)}\right] \boldsymbol{H}^{-1}, \quad u \geq 0. \end{aligned} \tag{3.31}$$

Now we can obtain $\mathbf{v}_k(u)$, for $k = 1, \cdots, n+1$, by using the same argument in the evaluation of $\mathbf{v}(u)$. Let

$$\boldsymbol{L}_{c_k}(s) = \left(s - \frac{\delta}{c_k}\right)\boldsymbol{I} + \frac{1}{c_k}(\boldsymbol{D_0} + \boldsymbol{\Lambda}_{\hat{\boldsymbol{f}}}(s)), \tag{3.32}$$

where c_k is the premium rate for the kth layer. Similar to the case of $\det[\boldsymbol{L}_c(s)] = 0$ discussed in this section before, equation $\det[\boldsymbol{L}_{c_k}(s)] = 0$ has exactly m roots with positive real parts, say, $\rho_{1,k}, \rho_{2,k}, \cdots, \rho_{m,k}$. Then parallel to (3.29), for $k = 1, 2, \cdots, n+1$, we have

$$\mathbf{v}_k(u) = [e^{\boldsymbol{K}_k u} - \boldsymbol{\psi}_k(u)][\boldsymbol{I} - \boldsymbol{\psi}_k(0)]^{-1},$$

where $\boldsymbol{K}_k = \boldsymbol{H}_k \Delta_{\rho,k} \boldsymbol{H}_k^{-1}$, $\Delta_{\rho,k} = \mathrm{diag}[\rho_{1,k}, \cdots, \rho_{m,k}]$ and $\boldsymbol{H}_k = (\vec{h}_{1,k}, \vec{h}_{2,k}, \cdots, \vec{h}_{m,k})$ with column vector $\vec{h}_{i,k}$ being an eigenvector of $\boldsymbol{L}_{c,k}(\rho_{i,k})$ corresponding to eigenvalue 0. Furthermore similar to (3.31), we can write $\mathbf{v}_k(u)$ as

$$\mathbf{v}_k(u) = \boldsymbol{H}_k \,\mathrm{diag}\left[\frac{e^{\rho_{1,k}u} - \theta_{1,k}(u)}{1 - \theta_{1,k}(0)}, \cdots, \frac{e^{\rho_{m,k}u} - \theta_{m,k}(u)}{1 - \theta_{m,k}(0)}\right] \boldsymbol{H}_k^{-1}, \quad u \geq 0, \tag{3.33}$$

where $\theta_{i,k}(u)$ is defined as

$$\theta_{i,k}(u) = E[e^{-\delta\tau - \rho_{i,k}|U(\tau)|} I(\tau < \infty)|U(0) = u], \quad u \geq 0, i \in E. \tag{3.34}$$

3.4 Initial Value $\vec{\phi}_k(b_{k-1})$

It was shown in Badescu (2008) that the initial value $\vec{\phi}(0)$ in (3.12) can be obtained by finding the left linearly independent row vectors, $\vec{q}_i^\top$, for each root ρ_i in $\det[\boldsymbol{L}_c(s)] = 0$ such that

$$\boldsymbol{Q}\left[(s - \frac{\delta}{c})\boldsymbol{I} + \frac{1}{c}(\boldsymbol{D_0} + \boldsymbol{\Lambda}_{\hat{\boldsymbol{f}}}(s))\right] = \boldsymbol{0}, \tag{3.35}$$

where $\boldsymbol{Q} = (\vec{q}_1, \cdots, \vec{q}_m)^\top$ is invertible. Using (3.13) and (3.35), $\vec{\phi}(0)$ can be further written as

$$\vec{\phi}(0) = \frac{1}{c}\sum_{i=1}^{m} \boldsymbol{Q}^{-1}\,\mathrm{diag}[\hat{\zeta}_i(\rho_1), \cdots, \hat{\zeta}_i(\rho_m)]\boldsymbol{Q}\vec{e}_i, \tag{3.36}$$

where $\hat{\zeta}_i(s)$ is the ith element of column vector $\mathcal{L}_s[\vec{\zeta}(u)]$ and $\vec{e}_i$ corresponds to the ith column vector of the identity matrix. Hence, equations (3.11), (3.12) and (3.36) complete the result of Lemma 1.

To find the expression for $\vec{\phi}_k(b_{k-1})$ in (3.15), we need to apply the Dickson-Hipp operator T_r instead of the ordinary Laplace transform operator for the case that $u \geq b_{k-1}$. The Dickson-Hipp transformation with the operator T_r for an integrable function f with respect to a real number r was defined in Dickson and Hipp (2001) as

$$T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u)du, \quad x \geq 0.$$

For vector function $\vec{W}(y)$ with each element being integrable function of y, define

$$T_r\vec{W}(y) = \int_y^\infty e^{-r(u-y)}\vec{W}(u)du, \quad y \geq 0.$$

Then

$$T_{r_1}T_{r_2}\vec{W}(y) = T_{r_2}T_{r_1}\vec{W}(y) = \frac{T_{r_1}\vec{W}(y) - T_{r_2}\vec{W}(y)}{r_2 - r_1}, \quad r_1 \neq r_2, \quad y \geq 0.$$

Using an approach which is similar to the one used in Lu and Li (2009a) for deriving $\vec{\phi}_2(b)$ in Markov-modulated risk model and in Lu and Li (2009b) for deriving $\vec{g}(b_{k-1})$ in the Sparre Andersen risk model, for $u \geq b_{k-1}$, we take Dickson-Hipp transform of both sides of (3.14). That is to multiply both sides of equation by $e^{-s(u-b_{k-1})}$ and to integrate with respect to u from b_{k-1} to ∞, then we can obtain

$$\boldsymbol{L}_{c_k}(s)T_s\vec{\phi}(b_{k-1}) = \vec{\phi}(b_{k-1}) - \frac{1}{c_k}T_s\vec{\zeta}_k(b_{k-1}), \quad u \geq b_{k-1}. \tag{3.37}$$

For each root in $\det[\boldsymbol{L}_{c_k}(s)] = 0$ in the kth threshold $\rho_{i,k}$, $i = 1, 2, \cdots, m$, letting $s = \rho_{i,k}$, we can find the linearly independent vector $\vec{q}_{i,k}$ to eigenvalue 0 such that $\vec{q}_{i,k}^{\top}\boldsymbol{L}_{c_k}(\rho_{i,k}) = 0$. Left-multiplying both sides of (3.37) yields m equations

$$0 = \vec{q}_{i,k}^{\top}\boldsymbol{L}_{c_k}(\rho_{i,k})T_{\rho_{i,k}}\vec{\phi}_k(b_{k-1}) = \vec{q}_{i,k}^{\top}\left(\vec{\phi}_k(b_{k-1}) - \frac{1}{c_k}T_{\rho_{i,k}}\vec{\zeta}_k(b_{k-1})\right).$$

In matrix form,

$$\boldsymbol{Q}_k\vec{\phi}_k(b_{k-1}; B) = \frac{1}{c_k}\sum_{i=1}^{m} \operatorname{diag}\left[T_{\rho_{1,k}}\vec{\zeta}_k(b_{k-1}), \cdots, T_{\rho_{m,k}}\vec{\zeta}_k(b_{k-1})\right]\boldsymbol{Q}_k\vec{e}_i,$$

where $\boldsymbol{Q}_k = [\vec{q}_{1,k}, \cdots, \vec{q}_{m,k}]^{\top}$ denotes the matrix with $\vec{q}_{i,k}^{\top}$ in its ith row. Then $\boldsymbol{Q}$ is invertible and

$$\vec{\phi}_k(b_{k-1}) = \frac{1}{c_k}\sum_{i=1}^{m} \boldsymbol{Q}_k^{-1} \operatorname{diag}\left[T_{\rho_{1,k}}\vec{\zeta}_k(b_{k-1}), \cdots, T_{\rho_{m,k}}\vec{\zeta}_k(b_{k-1})\right]\boldsymbol{Q}_k\vec{e}_i, \tag{3.38}$$

which completes the result presented in Lemma 2.

Chapter 4

Expected Discounted Penalty Function

In this chapter, we study the expected discounted penalty function in the multi-threshold MAP risk model by using the results obtained in the previous chapter. Recall the expected discounted penalty function in the multi-threshold MAP risk model defined in (3.2) in Chapter 3. It is a piecewise-defined vector of functions in the form of

$$\phi_i(u;B) = \begin{cases} \phi_{i,1}(u;B) & 0 \le u < b_1, \\ \phi_{i,k}(u;B) & b_{k-1} \le u < b_k, \quad k = 2, \cdots, n, \\ \phi_{i,n+1}(u;B) & b_n \le u < \infty. \end{cases} \tag{4.1}$$

For $k = 1, \cdots, n+1$, define a new vector of functions

$$\vec{\phi}_k(u;B) = (\phi_{1,k}(u;B), \cdots, \phi_{m,k}(u;B))^\top, \quad b_{k-1} \le u < b_k.$$

Then the expected discounted penalty function in the multi-threshold MAP model, $\vec{\phi}(u;B)$, is a piecewise-defined vector of functions in the form of

$$\vec{\phi}(u;B) = \begin{cases} \vec{\phi}_1(u;B) & 0 \le u < b_1, \\ \vec{\phi}_k(u;B) & b_{k-1} \le u < b_k, \quad k = 2, \cdots, n, \\ \vec{\phi}_{n+1}(u;B) & b_n \le u < \infty. \end{cases} \tag{4.2}$$

4.1 Piecewise Integro-differential Equation for $\vec{\phi}_k(u;B)$

Badescu (2008) presented a method of deriving a system of integro-differential equations for the MAP risk model with no dividend strategy involved. By similar arguments used there, for $b_{k-1} \le u < b_k$, $i \in E$ and $\delta > 0$, in an infinitesimal time interval $[0,h]$, we consider four scenarios as follows:

- no change in the MAP phase;
- a change in the MAP phase accompanied by no claim arrival;
- a change in the MAP phase accompanied by a claim arrival;
- two or more events occur.

When there is a claim arrival, the claim amount may vary, leading to the surplus process dropping below level 0 (ruin occurs) or starting again from the lth threshold interval, $l = 1, \cdots, k$. Mathematically, this can be written as

$$\begin{aligned}\phi_{i,k}(u;B) \quad = \quad & (1+D_0(i,i)h)e^{-\delta h}\phi_{i,k}(u+ch;B) + \sum_{j=1,j\neq i}^{m} D_0(i,j)he^{-\delta h}\phi_{j,k}(u+ch;B) \\ & + \sum_{j=1}^{m} D_1(i,j)he^{-\delta h}\int_0^{u-b_{k-1}+c_k h} \phi_{j,k}(u+c_k h - x;B)dF_{i,j}(x) \\ & + \sum_{j=1}^{m} D_1(i,j)he^{-\delta h}\left[\sum_{l=1}^{k-1}\int_{u-b_l+c_k h}^{u-b_{l-1}+c_k h} \phi_{j,l}(u+c_k h - x;B)dF_{i,j}(x) + \omega_{i,j}(u)\right] \\ & + o(h), \quad b_{k-1} \leq u < b_k, i \in E, \qquad (4.3)\end{aligned}$$

where $\omega_{i,j}(u)$ is given by (3.5). The first term on the right side of (4.3) corresponds to the case where no claim and no change of the state occur between time t and $t+h$. The second term corresponds to the case where the state changes but no claim. The third term corresponds to the case where the state changes with a claim, in which the claim amount is small enough such that the surplus process restarts from a new initial surplus without dropping below b_{k-1}. The forth term corresponds to the case that is similar to the third term but with different claim amounts. In this case, the surplus process restarts from the lower layers. The fifth term corresponds the case where ruin occurs and the penalty function is applied. The last term corresponds to the case where two or more events occur. All of the cases above are discounted by the valuation discount factor δ.

Note that for $i \in E$ and $k = 1, \cdots, n+1$, Taylor expansions give

$$\phi_{i,k}(u+ch;B) = \phi_{i,k}(u;B) + \phi'_{i,k}(u;B)ch + o(h),$$

and

$$e^{-\delta h} = 1 - \delta h + o(h).$$

Then we have

$$\begin{aligned}
\phi_{i,k}(u;B) &= \phi_{i,k}(u;B) + \phi'_{i,k}(u;B)ch - \delta h \phi_{i,k}(u+ch;B) \\
&+ \sum_{j=1}^{m} D_0(i,j)h(1-\delta h + o(h))\phi_{j,k}(u+ch;B) \\
&+ \sum_{j=1}^{m} D_1(i,j)h(1-\delta h + o(h)) \int_0^{u-b_{k-1}+c_k h} \phi_{j,k}(u+c_k h - x;B)dF_{i,j}(x) \\
&+ \sum_{j=1}^{m} D_1(i,j)h(1-\delta h + o(h)) \\
&\left[\sum_{l=1}^{k-1} \int_{u-b_l+c_k h}^{u-b_{l-1}+c_k h} \phi_{j,l}(u+c_k h - x;B)dF_{i,j}(x) + \omega_{i,j,k}(u)\right] \\
&+ o(h), \quad b_{k-1} \le u < b_k, i \in E. \qquad (4.4)
\end{aligned}$$

By dividing by h on both sides of (4.4) and letting $h \to 0$, it follows that

$$\begin{aligned}
c_k \phi'_{i,k}(u;B) &= \delta \phi_{i,k}(u;B) - \sum_{j=1}^{m} D_0(i,j)\phi_{j,k}(u;B) \\
&- \sum_{j=1}^{m} D_1(i,j) \int_0^{u-b_{k-1}} \phi_{j,k}(u-x;B)dF_{i,j}(x) \\
&- \xi_{i,k}(u), \quad b_{k-1} < u < b_k, i \in E, \qquad (4.5)
\end{aligned}$$

where

$$\xi_{i,k}(u) = \sum_{j=1}^{m} D_1(i,j) \left[\sum_{l=1}^{k-1} \int_{u-b_l}^{u-b_{l-1}} \phi_{j,l}(u-x;B)dF_{i,j}(x) + \omega_{i,j,k}(u)\right]. \qquad (4.6)$$

Note that $\xi_{i,k}(u)$ involves the evaluation of the expected discounted penalty functions from all of the lower layers. We do not include the case that $u = b_{k-1}$ in equation (4.5). As we will see later in this section, the derivative at each threshold does not exist. In the sequel, the same argument is applied to all the cases involving derivatives at thresholds. In matrix form, we have the following expression for $\vec{\phi}_k(u;B)$ when $b_{k-1} < u < b_k$:

$$c_k \vec{\phi}'_k(u;B) = \delta \vec{\phi}_k(u;B) - \boldsymbol{D_0} \vec{\phi}_k(u;B) - \int_0^{u-b_{k-1}} \boldsymbol{\Lambda_f}(x) \vec{\phi}_k(u-x;B)dx - \vec{\xi}_k(u), \qquad (4.7)$$

where $\vec{\xi}_k(u) = (\xi_{1,k}(u), \cdots, \xi_{m,k}(u))^\top$ with the ith element given by (4.14). In the case that $c_k = c$, $k = 1, \cdots, n+1$, the model reduces to the MAP risk model with no dividend strategy involved. Then equation (4.5) reduces to the equation presented in Badescu (2008).

The continuity conditions for the set of vectors $\vec{\phi}_1(u;B), \cdots, \vec{\phi}_{n+1}(u;B)$ are

$$\vec{\phi}_k(b_k-;B) = \vec{\phi}_{k+1}(b_k+;B), \quad k = 1, \cdots, n. \qquad (4.8)$$

It is also interesting to see that the derivatives of these vectors are not continuous at thresholds. From (4.7), we have

$$\begin{aligned} c_k\vec{\phi}_k'(b_k-;B) &= \delta\vec{\phi}_k(b_k-;B) - \boldsymbol{D_0}\vec{\phi}_k(b_k-;B) \\ &\quad - \int_0^{(b_k-)-b_{k-1}} \boldsymbol{\Lambda_f}(x)\vec{\phi}_k(u-x;B)dx - \vec{\xi}_k(b_k-) \\ &= \delta\vec{\phi}_{k+1}(b_k+;B) - \boldsymbol{D_0}\vec{\phi}_{k+1}(b_k+;B) - \vec{\xi}_{k+1}(b_k+) \\ &= c_{k+1}\vec{\phi}_{k+1}'(b_k+;B). \end{aligned}$$

4.2 Analytical Expression for $\vec{\phi}_k(u;B)$

In this section, we derive an analytical expression for $\vec{\phi}_k(u;B)$ by using the result in Lemma 2.

First letting $\vec{\zeta}_k(u) = \vec{\xi}_k(u)$ in Lemma 2 and relaxing the restriction $b_{k-1} < u < b_k$ to $u \geq b_{k-1}$, $\vec{\phi}_k(u;B)$ in (4.7) satisfies the non-homogeneous integro-differential equation (3.14). Then its solution is given by Lemma 2, with restriction $b_{k-1} \leq u < b_k$, as follows,

$$\vec{\phi}_k(u;B) = \mathbf{v}_k(u-b_{k-1})\vec{\phi}_k(b_{k-1};B) + \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\xi}_k(u-t)dt, \quad b_{k-1} \leq u < b_k, \tag{4.9}$$

where $\mathbf{v}_k(u-b_{k-1})$ is given by (3.16). Equation (4.9) holds for the case that $u = b_{k-1}$ because of the continuity condition in (4.8).

Note that in (4.9), $\vec{\phi}_k(b_{k-1};B)$ is unknown. In order to get an analytical expression for the piecewise vector function $\vec{\phi}(u;B)$ in (4.2), we need to determine $\vec{\phi}_k(b_{k-1};B)$ for $k = 1, \cdots, n+1$. The method presented in Section 3.4 is not applicable here because of the domain restriction. We consider another vector function $\vec{\phi}_k(u)$ defined on $[b_{k-1}, \infty)$ which also satisfies the non-homogeneous integro-differential equation (3.14) with the same non-homogeneous term setting $\vec{\zeta}_k(u) = \vec{\xi}_k(u)$ in (4.14). Its solution is given by Lemma 2 as follows,

$$\vec{\phi}_k(u) = \mathbf{v}_k(u-b_{k-1})\vec{\phi}_k(b_{k-1}) - \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\xi}_k(u-t)dt, \quad u \geq b_{k-1}, \tag{4.10}$$

where $\mathbf{v}_k(u-b_{k-1})$ is given by (3.16) and $\vec{\phi}_k(b_{k-1})$ is given by (3.38) with $\vec{\zeta}_k(u) = \vec{\xi}_k(u)$, or precisely,

$$\vec{\phi}_k(b_{k-1}) = \frac{1}{c_k}\sum_{i=1}^{m} \boldsymbol{Q}_k^{-1}\,\mathrm{diag}[T_{\rho_{1,k}}\vec{\xi}_k(b_{k-1}), \cdots, T_{\rho_{m,k}}\vec{\xi}_k(b_{k-1})]\boldsymbol{Q}_k\vec{e}_i, \tag{4.11}$$

where $\boldsymbol{Q}_k = (\vec{q}_{1,k}, \cdots, \vec{q}_{m,k})^\top$ and $\vec{q}_{i,k}^\top$ is the left eigenvector to eigenvalue 0 of matrix $\boldsymbol{L}_{c_k}(\rho_{i,k})$.

4.3 Recursive Expression for $\vec{\phi}_k(u;B)$

Restricting equation (4.10) to $b_{k-1} \le u \le b_k$, it is observed that the second term on the right hand sides of equation (4.9) and (4.10) are exactly the same. In this section, we rely on the relationship between (4.9) and (4.10) to develop a recursive algorithm to determine $\vec{\phi}_k(b_{k-1};B)$ and then to complete an analytical expression for $\vec{\phi}_k(u;B)$, for $k = 1, \cdots, n+1$.

Subtracting (4.10) from (4.9), we can rewrite (4.9) as

$$\begin{aligned} \vec{\phi}_k(u;B) &= \vec{\phi}_k(u) + \mathbf{v}_k(u-b_{k-1})[\vec{\phi}_k(b_{k-1};B) - \vec{\phi}_k(b_{k-1})] \\ &= \vec{\phi}_k(u) + \mathbf{v}_k(u-b_{k-1})\vec{\kappa}_k(B), \quad b_{k-1} \le u < b_k, \end{aligned} \tag{4.12}$$

where the constant vector $\vec{\kappa}_k(B)$ is to be determined.

It follows from the continuity condition (4.8) at threshold b_{k-1} that

$$\vec{\phi}_k(b_k) + \mathbf{v}_k(b_k - b_{k-1})\vec{\kappa}_k(B) = \vec{\phi}_{k+1}(b_k) + \mathbf{v}_{k+1}(0)\vec{\kappa}_{k+1}(B), \quad k = 1, \cdots, n.$$

Since $\mathbf{v}_{k+1}(0) = I$, we obtain

$$\vec{\kappa}_{k+1}(B) = \vec{\phi}_k(b_k) - \vec{\phi}_{k+1}(b_k) + \mathbf{v}_k(b_k - b_{k-1})\vec{\kappa}_k(B), \quad k = 1, \cdots, n. \tag{4.13}$$

When $k = n+1$, $\vec{\phi}_{n+1}(u;B)$ satisfies an integro-differential equation for $u \ge b_n$. Equation (4.9) turns out to be exactly the same as equation (4.10). Thus we have $\vec{\phi}_{n+1}(u;B) = \vec{\phi}_{n+1}(u)$ and the boundary condition $\vec{\kappa}_{n+1}(B) = \vec{0}$ obtained from (4.13).

We now conclude our derived results for $\vec{\phi}_k(u;B)$ in the theorem below.

Theorem 1 *The analytical expression for the vector of expected discounted penalty functions $\vec{\phi}(u;B)$ can be obtained piecewisely as*

$$\vec{\phi}(u;B) = \vec{\phi}_k(u;B), \quad b_{k-1} \le u < b_k, k = 1, \cdots, n+1,$$

where $\vec{\phi}_k(u;B)$ is given by

$$\vec{\phi}_k(u;B) = \vec{\phi}_k(u) + \mathbf{v}_k(u-b_{k-1})\vec{\kappa}_k(B), \quad b_{k-1} \le u < b_k,$$

in which $\vec{\phi}_k(u)$ and $\vec{\kappa}_k(B)$ for $k = 1, \cdots, n+1$ is obtained recursively by

$$\vec{\phi}_k(u) = \mathbf{v}_k(u-b_{k-1})\vec{\phi}_k(b_{k-1}) - \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\xi}_k(u-t)dt, \quad b_{k-1} \le u < b_k,$$

and

$$\begin{cases} \vec{\kappa}_{k+1}(B) = \vec{\phi}_k(b_k) - \vec{\phi}_{k+1}(b_k) + \mathbf{v}_k(b_k - b_{k-1})\vec{\kappa}_k(B), & k = 1, \cdots, n, \\ \vec{\kappa}_{n+1}(B) = \vec{0}, \end{cases}$$

with the ith element of $\vec{\xi}_k(u)$ *being*

$$\xi_{i,k}(u) = \sum_{j=1}^{m} D_1(i,j)\left[\sum_{l=1}^{k-1}\int_{u-b_l}^{u-b_{l-1}} \phi_{j,l}(u-x;B)dF_{i,j}(x) + \omega_{i,j}(u)\right], \quad b_{k-1} \le u < b_k, \tag{4.14}$$

and

$$\omega_{i,j}(u) = \int_u^\infty w(u, x-u)dF_{i,j}(x), \quad b_{k-1} \le u < b_k.$$

The analytical expressions of matrix function $\mathbf{v}_k(u)$ *and constant vector* $\vec{\phi}_k(b_{k-1})$ *are given by* (3.16) *and* (4.11)*, respectively.*

As remarked by Lin and Sendova (2008) for the classical compound Poisson risk model and Lu and Li (2009b) for the Sparre Andersen risk model, the recursive scheme provided by Theorem 1 is not quite obvious to be implemented. Here we provide an algorithm based on Theorem 1 for obtaining the analytical expression $\vec{\phi}(u;B)$ in (4.2).

Algorithm:

1. For each c_k, $k = 1, \cdots, n+1$, find corresponding m solutions $\rho_{i,k}$, $i = 1, \cdots, m$ to equation $\det[\boldsymbol{L}_{c_k}(s)] = 0$, where $\boldsymbol{L}_{c_k}(s)$ is given by (3.32). Then obtain $\boldsymbol{Q}_k = (\vec{q}_{1,k}, \cdots, \vec{q}_{m,k})^\top$ with $\vec{q}_{i,k}^{\,\top}$ being the left eigenvector to the eigenvalue 0 of matrix $\boldsymbol{L}_{c_k}(\rho_{i,k})$, and $\boldsymbol{H}_k = (\vec{h}_{1,k}, \cdots, \vec{h}_{m,k})$ with $\vec{h}_{i,k}$ being the right eigenvector to the eigenvalue 0 of matrix $\boldsymbol{L}_{c_k}(\rho_{i,k})$.

2. For each root $\rho_{i,k}$, $i = 1, \cdots, m$ and $k = 1, \cdots, n+1$, obtain the expression of $\theta_{i,k}(u)$ given by (3.34), which is a special case of the expected discounted penalty function with $w(x,y) = e^{-\rho_{i,k}y}$ in the MAP risk model without dividend strategy. The expression of matrix function $\mathbf{v}_k(u)$, $k = 1, \cdots, n+1$, can be obtained by equation (3.33).

3. For $k = 1$, we have $\vec{\xi}_1(u) = (\xi_{1,1}(u), \cdots, \xi_{m,1}(u))^\top$ with

$$\xi_{i,1}(u) = \sum_{j=1}^{m} D_1(i,j)\omega_{i,j}(u),$$

by equation (4.14),where $\omega_{i,j}(u) = \int_u^\infty w(u, x-u)dF_{i,j}(x)$. Using Dickson-Hipp operator and matrix $\boldsymbol{Q}_1$ obtained in step 1, we can get

$$\vec{\phi}_1(0) = \frac{1}{c_1}\sum_{i=1}^{m} \boldsymbol{Q}_1^{-1}\,\mathrm{diag}[T_{\rho_{1,1}}\vec{\xi}_1(0), \cdots, T_{\rho_{m,1}}\vec{\xi}_1(0)]\boldsymbol{Q}_1\vec{e}_i,$$

by equation (4.11) and

$$\vec{\phi}_1(u) = \mathbf{v}_1(u)\vec{\phi}_1(0) - \frac{1}{c_1}\int_0^u \mathbf{v}_1(t)\vec{\xi}_1(u-t)dt, \quad u \ge 0,$$

by equation (4.10).

4. Restricting $\vec{\phi}_1(u)$ to $0 \le u < b_1$, then $\vec{\phi}_1(u;B)$ can be obtained by (4.12) as

$$\vec{\phi}_1(u;B) = \vec{\phi}_1(u) + \mathbf{v}_1(u)\vec{\kappa}_1(B), \quad 0 \le u < b_1,$$

where $\mathbf{v}_1(u)$ is obtained in step 2 and $\vec{\kappa}_1(B)$ is an unknown constant vector that will be determined in the last step of this algorithm. Note that the expression of $\vec{\phi}_1(u;B)$ for $0 \le u < b_1$ by now is a function of the unknown vector $\vec{\kappa}_1(B)$.

5. For $k = 2$, we have $\vec{\xi}_2(u) = (\xi_{1,2}(u), \cdots, \xi_{m,2}(u))^\top$ with

$$\xi_{i,2}(u) = \sum_{j=1}^{m} D_1(i,j) \int_{u-b_1}^{u} \phi_{j,1}(u-x;B)dF_{i,j}(x) + \xi_{i,1}(u),$$

where $\phi_{j,1}(u;B)$ is the jth element of $\vec{\phi}_1(u;B)$. By equation (4.11), (4.10) and $\boldsymbol{Q}_2$ obtained in step 1, we can get

$$\vec{\phi}_2(b_1) = \frac{1}{c_2} \sum_{i=1}^{m} \boldsymbol{Q}_2^{-1} \operatorname{diag}[T_{\rho_{1,2}}\vec{\xi}_2(b_1), \cdots, T_{\rho_{m,2}}\vec{\xi}_2(b_1)]\boldsymbol{Q}_2\vec{e}_i,$$

and

$$\vec{\phi}_2(u) = \mathbf{v}_2(u-b_1)\vec{\phi}_2(b_1) - \frac{1}{c_2}\int_0^{u-b_1} \mathbf{v}_2(t)\vec{\xi}_2(u-t)dt, \quad u \ge b_1.$$

Note that $\vec{\xi}_2(u)$ is a function of $\vec{\kappa}_1(B)$. As a consequence, both $\vec{\phi}_2(b_1)$ and $\vec{\phi}_2(u)$ are functions of $\vec{\kappa}_1(B)$.

6. Restricting $\vec{\phi}_2(u)$ to $b_1 \le u < b_2$, we have

$$\vec{\phi}_2(u;B) = \vec{\phi}_2(u) + \mathbf{v}_2(u-b_1)\vec{\kappa}_2(B), \quad b_1 \le u < b_2,$$

where $\mathbf{v}_2(u-b_1)$ is obtained in step 2 and

$$\vec{\kappa}_2(B) = \vec{\phi}_1(b_1) - \vec{\phi}_2(b_1) + \mathbf{v}_1(b_1)\vec{\kappa}_1(B).$$

The relationship between $\vec{\kappa}_1(B)$ and $\vec{\kappa}_2(B)$ implies that $\vec{\kappa}_2(B)$ and $\vec{\phi}_2(u;B)$ for $b_1 \le u < b_2$ are also functions of $\vec{\kappa}_1(B)$.

7. Similarly, for $k = 3, \cdots, n$, we have

$$\xi_{i,k}(u) = \sum_{j=1}^{m} D_1(i,j) \int_{u-b_{k-1}}^{u-b_{k-2}} \phi_{j,k-1}(u-x;B)dF_{i,j}(x) + \xi_{i,k-1}(u).$$

Again using $\boldsymbol{Q}_k$ obtained in step 1, we can get $\vec{\phi}_k(b_{k-1})$ by equation (4.11) and $\vec{\phi}_k(u;B)$ by (4.10) for $u \ge b_{k-1}$, and then

$$\vec{\phi}_k(u;B) = \vec{\phi}_k(u) + \mathbf{v}_k(u-b_{k-1})\vec{\kappa}_k(B),$$

where $\mathbf{v}_k(u - b_{k-1})$ is obtained in step 2 and

$$\vec{\kappa}_k(B) = \vec{\phi}_{k-1}(b_{k-1}) - \vec{\phi}_k(b_{k-1}) + \mathbf{v}_{k-1}(b_{k-1} - b_{k-2})\vec{\kappa}_{k-1}(B).$$

By the same argument in step 6, we conclude here that the expression of $\vec{\kappa}_{k-1}(B)$ and $\vec{\phi}_k(u; B)$ for $b_{k-1} \leq u < b_k$ are functions of $\vec{\kappa}_1(B)$, where $k = 3, \cdots, n$.

8. For $k = n + 1$, the expression of $\vec{\xi}_{n+1}(u)$ involves all the previous vector function $\vec{\phi}_k(u; B)$, $k = 1, \cdots, n$. In the last layer, we have $\vec{\phi}_{n+1}(u; B) = \vec{\phi}_{n+1}(u)$ for $u \geq b_n$, and then $\vec{\kappa}_{n+1}(B) = \vec{0}$. Note that $\vec{\phi}_{n+1}(b_n)$, $\vec{\phi}_{n+1}(u)$, $\vec{\phi}_{n+1}(u; B)$ and $\vec{\kappa}_n(B)$ are all the functions of $\vec{\kappa}_1(B)$. Finally the unknown vector $\vec{\kappa}_1(B)$ can be obtained by solving

$$\vec{0} = \vec{\kappa}_{n+1}(B) = \vec{\phi}_n(b_n) - \vec{\phi}_{n+1}(b_n) + \mathbf{v}_n(b_n - b_{n-1})\vec{\kappa}_n(B)$$

which completes all the calculations. We have all the vector functions needed to obtain an analytical expression of the piecewise-defined vector of functions $\vec{\phi}(u; B)$ in (4.2).

Chapter 5

Expected Discounted Dividend Payments

As remarked in Chapter 1, though the expression of the expected discounted dividend payments is not a special case of the expected discounted penalty function, most techniques used in Chapter 4 can be adopted to obtaining the expected discounted dividend payments, involving a system of integro-differential equations and constant vectors. The purpose of this chapter is to study the expected discounted dividend payments in the multi-threshold MAP risk model. Define

$$D_{u,B} = \int_0^{\tau_B} e^{\delta t} dD(t), \quad u \geq 0, \tag{5.1}$$

to be the present value of all dividends until the time of ruin τ_B given the initial surplus u in the multi-threshold MAP risk model, where $D(t)$ is the aggregate dividends paid by time t. Let

$$V_i(u; B) = \mathbb{E}_i[D_{u,B} | U_B(0) = u], \quad u \geq 0, i \in E, \tag{5.2}$$

be the expected present value of the dividend payments before the time of ruin under a multi-threshold dividend strategy, given the initial surplus u and the initial phase $i \in E$. In terms of vector, the expected present value of the dividend payments prior to ruin is $\vec{V}(u; B) = (V_1(u; B), \cdots, V_m(u; B))^\top$. Similar to the piecewise function $\phi_i(u; B)$ defined in Chapter 4, the piecewise function of the expected present value of the total dividend payments prior to ruin given the initial phase i is defined for $u \geq 0$ as,

$$V_i(u; B) = \begin{cases} V_{i,1}(u; B) & 0 \leq u < b_1, \\ V_{i,k}(u; B) & b_{k-1} \leq u < b_k, \quad k = 2, \cdots, n, \\ V_{i,n+1}(u; B) & b_n \leq u < \infty, \end{cases} \tag{5.3}$$

and the piecewise vector function of the expected present value of the total dividend payments prior to ruin is defined for $u \geq 0$ by,

$$\vec{V}(u;B) = \begin{cases} \vec{V}_1(u;B) & 0 \leq u < b_1, \\ \vec{V}_k(u;B) & b_{k-1} \leq u < b_k, \quad k = 2, \cdots, n, \\ \vec{V}_{n+1}(u;B) & b_n \leq u < \infty, \end{cases} \tag{5.4}$$

where $\vec{V}_k(u;B) = (V_{1,k}(u;B), \cdots, V_{m,k}(u;B))^\top$ for $b_{k-1} \leq u < b_k$ and $k = 1, \cdots, n+1$.

In the case of the MAP risk model with a constant barrier b, the surplus process (3.1) is modified to $U_b(t)$, where the initial surplus $U_b(0) = u$. Similar to the multi-threshold MAP risk model, we define $\tau_b = \inf\{t \geq 0 : U_b(t) < 0\}$ to be the time of ruin and

$$D_{u,b} = \int_0^{\tau_b} e^{-\delta t} dD(t), \quad 0 \leq u \leq b, \tag{5.5}$$

to be the present value of the dividend payments prior to ruin given that the initial surplus u in the MAP risk model with a constant barrier strategy. Then define

$$V_i(u;b) = \mathbb{E}_i[D_{u,b} | U_b(0) = u], \quad 0 \leq u \leq b, i \in E, \tag{5.6}$$

to be the expected present value of the dividend payments prior to ruin given the initial phase i the the initial surplus u. Later in this chapter, we will see that the expression of the expected present values of the dividend payments in the multi-threshold MAP risk model is related to the one in the MAP risk model with a constant barrier strategy.

5.1 Piecewise Integro-differential Equation for $\vec{V}_k(u;B)$

By similar arguments used in deriving the integro-differential equations for $V(u;b)$ in Chapter 4, we derive an integro-differential equation for $\vec{V}_k(u;B)$. For $b_{k-1} \leq u < b_k$, $i \in E$ and $\delta > 0$, conditioning on the events occurring in an infinitesimal time interval $[0,h]$, we have

$$\begin{aligned} V_{i,k}(u;B) \quad = \quad & (c - c_k)h + (1 + D_0(i,i)h)e^{-\delta h} V_{i,k}(u+ch;B) \\ & + \sum_{j=1, j\neq i}^{m} D_0(i,j)he^{-\delta h} V_{j,k}(u+ch;B) \\ & + \sum_{j=1}^{m} D_1(i,j)he^{-\delta h} \int_0^{u-b_{k-1}+c_k h} V_{j,k}(u+c_k h - x;B)dF_{i,j}(x) \\ & + \sum_{j=1}^{m} D_1(i,j)he^{-\delta h} \left[\sum_{l=1}^{k-1} \int_{u-b_l+c_k h}^{u-b_{l-1}+c_k h} V_{j,l}(u+c_k h - x;B)dF_{i,j}(x) \right] \\ & + o(h), \quad b_{k-1} \leq u < b_k, i \in E. \end{aligned} \tag{5.7}$$

Applying Taylor expansion, dividing by h on both sides of (5.7) and letting $h \to 0$, we get

$$\begin{aligned} c_k V'_{i,k}(u;B) &= \delta V_{i,k}(u;B) - \sum_{j=1}^{m} D_0(i,j) V_{j,k}(u;B) \\ &\quad - \sum_{j=1}^{m} D_1(i,j) \int_0^{u-b_{k-1}} V_{j,k}(u-x;B) dF_{i,j}(x) \\ &\quad -\gamma_{i,k}(u), \quad b_{k-1} \le u < b_k, i \in E, \end{aligned} \tag{5.8}$$

where

$$\gamma_{i,k}(u) = (c - c_k) + \sum_{j=1}^{m} D_1(i,j) \sum_{l=1}^{k-1} \int_{u-b_l}^{u-b_{l-1}} V_{j,l}(u-x;B) dF_{i,j}(x). \tag{5.9}$$

Note that the expression of $\gamma_{i,k}(u)$, which is the non-homogeneous term in equation (5.8), involves the expected discounted dividend payments in the lower layers and a constant difference between premium rates in the kth layer and in the first layer. Then we obtain the following expression in matrix form for $\vec{V}_k(u;B)$ as

$$\begin{aligned} c_k \vec{V}'_k(u;B) &= \delta \vec{V}_k(u;B) - \boldsymbol{D_0} \vec{V}_k(u;B) \\ &\quad - \int_0^{u-b_{k-1}} \boldsymbol{\Lambda_f}(x) \vec{V}_k(u-x;B) dx - \vec{\gamma}_k(u), \quad b_{k-1} \le u < b_k, \end{aligned} \tag{5.10}$$

where $\vec{\gamma}_k(u) = (\gamma_{1,k}(u), \cdots, \gamma_{m,k}(u))^\top$. The continuity conditions for the set of vectors $\vec{V}_1(u;B), \cdots, \vec{V}_{n+1}(u;B)$ are

$$\vec{V}_k(b_k-;B) = \vec{V}_{k+1}(b_k+;B), \quad k = 1, \cdots, n. \tag{5.11}$$

It is interesting to note that the derivative of $\vec{V}_k(u;B)$ is not continuous at each threshold level. It has the following relationship:

$$c_k \frac{d\vec{V}_k(u;B)}{du}\Big|_{u=b_k-} = c_{k+1} \frac{d\vec{V}_{k+1}(u;B)}{du}\Big|_{u=b_k+} + (c - c_{k+1}) \vec{V}_{k+1}(b_k+;B).$$

5.2 Analytical Expression for $\vec{V}_k(u;B)$

Relaxing the integro-differential equation (5.10) to $u \ge b_{k-1}$ and comparing it with (3.14), we observe that the only difference is the non-homogeneous term; instead of $\vec{\zeta}_k(u)$ in (3.14), we have $\vec{\gamma}_k(u)$ in (5.10). By applying Lemma 2 and then restricting to $b_{k-1} \le u < b_k$, its solution can be expressed as

$$\vec{V}_k(u;B) = \mathbf{v}_k(u-b_{k-1}) \vec{V}_k(b_{k-1};B) - \frac{1}{c_k} \int_0^{u-b_{k-1}} \mathbf{v}_k(t) \vec{\gamma}_k(u-t) dt, \quad b_{k-1} \le u < b_k, \tag{5.12}$$

where $\mathbf{v}_k(u)$ is given by (3.16).

In the case of the MAP risk model with a constant barrier $b = b_1$, when the surplus exceeds b_1, premiums collected thereafter are paid as dividends continuously. The surplus will stay at the level b_1 until a new claim occurs. For the case that $0 \leq u < b_1$, the second term in (5.9) disappears as no previous layer is involved and $c = c_1$. Then we have $\vec{\gamma}_1(u) = 0$. So that

$$c\vec{V}'(u;b) = \delta\vec{V}(u;b) - \boldsymbol{D_0}\vec{V}(u;b) - \int_0^u \boldsymbol{\Lambda_f}(x)\vec{V}(u-x;b)dx. \tag{5.13}$$

For the case that $u = b_1$, similarly conditioning on the event occurring in an infinitesimal time interval $[0,h]$, we have

$$c = \delta\vec{V}(b_1;b) - \boldsymbol{D_0}\vec{V}(b_1;b) - \int_0^u \boldsymbol{\Lambda_f}(x)\vec{V}(b_1-x;b)dx. \tag{5.14}$$

Setting $u = b_1-$ in (5.13), subtracting (5.14) from (5.13) and noting that $\vec{V}(u;b)$ is continuous at b gives immediately that $\vec{V}(u;b)$ satisfies condition $\vec{V}'(b_1-;b) = \vec{1}$. For the case that $u > b$, dividend $u-b$ is paid immediately, so $\vec{V}(u;b) = \vec{V}(b_1;b) + u - b_1$ and $\vec{V}'(b_1+;b) = \vec{1}$. Thus we have the following continuity condition for $\vec{V}'(u;b)$ at $u = b_1$:

$$\vec{V}'(b_1;b) = \vec{1}. \tag{5.15}$$

By using the same arguments in deriving $\vec{\phi}(u)$ in Section 3.1, we take Laplace transform of both sides of (5.13) and obtain

$$\vec{V}(u;b) = \mathbf{v}(u)\vec{V}(0;b), \quad 0 \leq u \leq b. \tag{5.16}$$

Further using the condition in (5.15), we have

$$\vec{V}'(u;b)\Big|_{u=b_1} = \mathbf{v}'(u)\Big|_{u=b_1}\vec{V}(0;b) = \vec{1}. \tag{5.17}$$

Then the initial value of the expected discounted dividend payments under a constant barrier strategy is given by

$$\vec{V}(0;b) = \left[\mathbf{v}'(b_1)\right]^{-1}\vec{1}. \tag{5.18}$$

When $k = 1$, we can obtain $\mathbf{v}_1(u) = \mathbf{v}(u)$ and $\vec{V}_1(u;B)$ from (5.12) as

$$\vec{V}_1(u;B) = \mathbf{v}(u)\vec{V}_1(0;B), \quad 0 \leq u < b. \tag{5.19}$$

By (5.16), (5.18) and (5.19), it is observed that $\vec{V}_1(u;B)$ can be rewritten as

$$\vec{V}_1(u;B) = \vec{V}(u;b) + \mathbf{v}(u)\vec{\pi}_1(B), \quad 0 \leq u < b, \tag{5.20}$$

where $\vec{\pi}_1(B) = \vec{V}_1(0;B) - \vec{V}(0;b) = \vec{V}_1(0;B) - [\mathbf{v}'(b)]^{-1}\vec{1}$. With this relationship between $\vec{V}_1(u;B)$ and $\vec{V}(u;b)$, the expression of the expected discounted dividend payments for the first layer involving a constant vector $\vec{\pi}_1(B)$ can be obtained analytically as a starting point. In the next section, we will determine the expected discounted dividend payments for the upper layers recursively.

5.3 Recursive Expression for $\vec{V}_k(u;B)$

Recalling the idea in Section 4.2, to find an analytical expression for $\vec{V}_k(u;B), k=2,\cdots,n+1$, we define vector function $\vec{V}_k(u;b)$ to be the solution to (3.14) with $\vec{\gamma}_k(u)$ as the non-homogeneous term. By Lemma 2, the solution is obtained as

$$\vec{V}_k(u;b) = \mathbf{v}_k(u-b_{k-1})\vec{V}_k(b_{k-1};b) - \frac{1}{c_k}\int_0^{u-b_{k-1}} \mathbf{v}_k(t)\vec{\gamma}_k(u-t)dt, \quad u \geq b_{k-1}, \tag{5.21}$$

where $\vec{V}_k(b_{k-1};b)$ can be obtained by the same arguments in Section 3.4 with the help of matrix $\boldsymbol{Q}_k$ and the Dickson-Hipp operators as

$$\vec{V}_k(b_{k-1};b) = \frac{1}{c_k}\sum_{i=1}^{m} \boldsymbol{Q}_k^{-1}\,\mathrm{diag}[T_{\rho_{1,k}}\vec{\gamma}_k(b_{k-1}),\cdots,T_{\rho_{m,k}}\vec{\gamma}_k(b_{k-1})]\boldsymbol{Q}_k\vec{e}_i. \tag{5.22}$$

Now restricting $\vec{V}_k(u;b)$ in (5.21) to $b_{k-1}\leq u<b_k$ and comparing (5.21) and (5.12), we can rewrite (5.12) as

$$\vec{V}_k(u;B) = \vec{V}_k(u;b) + \mathbf{v}_k(u-b_{k-1})\vec{\pi}_k(B), \quad b_{k-1}\leq u<b_k, \tag{5.23}$$

where $\vec{\pi}_k(B) = \vec{V}_k(b_{k-1};B) - \vec{V}_k(b_{k-1};b)$.

It follows the continuity condition (5.11) at threshold b_{k-1} and $\mathbf{v}_{k+1}(0)=I$ that

$$\vec{\pi}_{k+1}(B) = \vec{V}_k(b_k;b) - \vec{V}_{k+1}(b_k;b) + \mathbf{v}_k(b_k-b_{k-1})\vec{\pi}_k(B), \quad k=1,\cdots,n \tag{5.24}$$

When $k=n+1$, the expressions in (5.12) and (5.21) are exactly the same for $u\geq b_n$. Thus the series of constant vectors $\vec{\pi}_k(B)$, $k=1,\cdots,n$, can be solved by the final boundary condition $\vec{\pi}_{n+1}(B)=\vec{0}$.

Now we write an algorithm for calculating all the vector functions $\vec{V}_k(u;B)$, $k=1,\cdots,n+1$, which are used to obtain an analytical expression of the piecewise-defined vector function $\vec{V}(u;B)$ in (5.4).

Algorithm:

1. Similar to Step 1 in Section 4.3, for each c_k, $k=1,\cdots,n+1$, we can get m solutions $\rho_{i,k}$, $i=1,\cdots,m$ to equation $\det[\boldsymbol{L}_{c_k}(s)]=0$ and matrices $\boldsymbol{Q}_k$ and $\boldsymbol{H}_k$, $k=1,\cdots,n+1$.

2. Similar to Step 2 in Section 4.3, obtain the expression of matrix function $\mathbf{v}_k(u)$, $k=1,\cdots,n+1$.

3. For $k=1$, we have $\vec{\gamma}_1(u)=\vec{0}$ and

$$\vec{V}(u;b) = \mathbf{v}_1(u)\vec{V}(0;b) = \mathbf{v}_1(u)\left[\mathbf{v}_1'(b_1)\right]^\top\vec{1}, \quad 0\leq u\leq b_1, \tag{5.25}$$

where $\mathbf{v}_1(u)$ is obtained in Step 2. Remark that the expression of $\vec{V}(u;b)$ only relies on the matrix function $\mathbf{v}_1(u)$, while the expression of $\vec{\phi}_1$ in Chapter 4 relies on the non-homogeneous term $\vec{\xi}_1(u)$ and the initial value $\vec{\phi}_1(0)$.

4. Similar to Step 4 in Section 4.3, we have

$$\vec{V}_1(u;B) = \vec{V}(u;b) + \mathbf{v}_1(u)\vec{\pi}_1(B), \quad 0 \le u < b_1, \tag{5.26}$$

where $\vec{\pi}_1(B) = \vec{V}_1(0;B) - [\mathbf{v}_1'(b_1)]^\top \vec{1}$ is an unknown constant vector by now and it will be determined in the last step of this algorithm.

5. For $k = 2, \cdots, n$, we have $\vec{\gamma}_k(u) = (\gamma_{1,k}(u), \cdots, \gamma_{m,k}(u))^\top$ with

$$\gamma_{i,k}(u) = (c - c_k) + \sum_{j=1}^{m} D_1(i,j) \sum_{l=1}^{k-1} \int_{u-b_{l-1}}^{u-b_{l-2}} V_{j,l}(u-x;B)dF_{i,j}(x). \tag{5.27}$$

Then we can get $\vec{V}_k(b_{k-1};b)$ by (5.22) and $\vec{V}_k(u;b)$ by (5.21) for $u \ge b_{k-1}$.

6. Restricting $\vec{V}_k(u;b)$ to $b_{k-1} \le u < b_k$, $k = 2, \cdots, n$, we have

$$\vec{V}_k(u;B) = \vec{V}_k(u;b) + \mathbf{v}_k(u - b_{k-1})\vec{\pi}_k(B),$$

where $\mathbf{v}_k(u - b_{k-1})$ is determined in step 2 and

$$\vec{\pi}_k(B) = \vec{V}_{k-1}(b_{k-1};b) - \vec{V}_k(b_{k-1};b) + \mathbf{v}_{k-1}(b_{k-1} - b_{k-2})\vec{\pi}_{k-1}(B).$$

7. For $k = n+1$, we have $\vec{V}_{n+1}(u;B) = \vec{V}_{n+1}(u;b)$, for $u \ge b_n$, and $\vec{\pi}_{n+1}(B) = \vec{0}$. Referring to the explanation in Section 4.3, the expressions of $\vec{V}_{n+1}(b_n;b)$, $\vec{V}_n(u;b)$, $\vec{V}_{n+1}(u;B)$ and $\vec{\gamma}_n(B)$ are all the functions of $\vec{\pi}_1(B)$. The unknown vector $\vec{\gamma}_1(B)$ can be obtained by solving

$$\vec{0} = \vec{\gamma}_{n+1}(B) = \vec{V}_n(b_n;b) - \vec{\phi}_{n+1}(b_n;b) + \mathbf{v}_n(b_n - b_{n-1})\vec{\gamma}_n(B),$$

which completes all the calculations. Now We have all the vector functions needed to obtain an analytical expression of the piecewise-defined vector of functions $\vec{V}(u;B)$ in (5.4).

5.4 Moment Generating Function of $D_{u,B}$ and Higher Moments

In this section, we employ similar techniques used in Section 5.1, 5.2 and 5.3 to study the moment generating function of the dividend payments in the multi-threshold MAP risk model. It is shown that the higher moments of the present value of all dividend payments prior to ruin satisfy a system of integro-differential equations.

Define

$$M_{i,k}(u,y;B) = \mathbb{E}_i[e^{yD_{u,B}}|U(0)=u], \quad b_{k-1} \le u < b_k, i \in E,$$

to be the moment generating function of $D_{u,B}$, given that the initial MAP phase i and the initial surplus u. Similar to (4.3) in Section 4.1 and (5.7) in Session 5.1, conditioning on the events that occur in an infinitesimal time interval $[0,h]$, we have

$$\begin{aligned} M_{i,k}(u,y;B) &= e^{(c-c_k)hy}\Bigg\{[1+D_0(i,i)h]\,M_{i,k}(u+c_kh, e^{-\delta h}y;B) \\ &+ \sum_{j=1,j\neq i}^{m} D_0(i,j)hM_{j,k}(u+c_kh, e^{-\delta h}y;B) \\ &+\sum_{j=1}^{m} D_1(i,j)h\Bigg[\int_0^{u-b_{k-1}+c_kh} M_{j,k}(u+c_kh-x, e^{-\delta h}y;B)dF_{i,j}(x) \\ &+\sum_{l=1}^{k-1}\int_{u-b_l+c_kh}^{u-b_{l-1}+c_kh} M_{j,l}(u+c_kh-x,e^{-\delta h}y;B)dF_{i,j}(x)+\int_{u+c_kh}^{\infty} dF_{i,j}(x)\Bigg] \\ &+o(h)\Bigg\}, \quad b_{k-1}\le u<b_k, i\in E. \end{aligned} \tag{5.28}$$

As done in Li and Lu (2007), Taylor expansions give

$$M_{i,k}(u+c_kh, e^{-\delta h}y;B) = M_{i,k}(u,y;B) + c_kh\frac{\partial M_{i,k}(u,y;B)}{\partial u} - \delta yh\frac{\partial M_{i,k}(u,y;B)}{\partial y} + o(h), \tag{5.29}$$

and

$$e^{xh} = 1 + xh + o(h). \tag{5.30}$$

Substituting (5.29) and (5.30) into (5.28), dividing both sides of (5.28) by h and letting $h \to 0$, we then obtain

$$\begin{aligned} 0 &= (c-c_k)yM_{i,k}(u,y;B) + c_k\frac{\partial M_{i,k}(u,y;B)}{\partial u} - \delta y\frac{\partial M_{i,k}(u,y;B)}{\partial y} \\ &+ \sum_{j=1,j\neq i}^{m} D_0(i,j)M_{j,k}(u,y;B) \\ &+\sum_{j=1}^{m} D_1(i,j)\Bigg[\int_0^{u-b_{k-1}} M_{j,k}(u-x,y;B)dF_{i,j}(x) \\ &+\sum_{l=1}^{k-1}\int_{u-b_l}^{u-b_{l-1}} M_{j,l}(u-x,y;B)dF_{i,j}(x)+\bar{F}_{i,j}(u)\Bigg], \quad b_{k-1}\le u<b_k, \end{aligned} \tag{5.31}$$

where $\bar{F}_{i,j} = 1 - F_{i,j}$ is the survival distribution of the claim amounts.

Define, for $k = 1, \cdots, n+1$,

$$V_{i,k}^n(u; B) = \mathbb{E}_i[D_{u,B}^n | U(0) = u], \quad b_{k-1} \leq u < b_k, i \in E,$$

to be the nth moment of $D_{u,B}$, given that the initial MAP phase i and the initial surplus u, with $V_{i,k}^0(u; B) = 1$ and $V_{i,k}^1(u; B) = V_{i,k}(u; B)$ studied in Section 5.1. Following the representation of moment generation function

$$M_{i,k}(u, y; B) = 1 + \sum_{n=1}^{\infty} (y^n/n!) V_{i,k}^n(u; B),$$

and comparing the coefficient of $y^n/n!$ in (5.31), we obtain, for $b_{k-1} < u < b_k$, that

$$\begin{aligned} & c_k \frac{dV_{i,k}^n(u;B)}{du} - n\delta V_{i,k}^n(u;B) + \sum_{j=1}^m D_0(i,j) V_{j,k}^n(u;B) + n(c-c_k) V_{i,k}^{n-1}(u;B) \\ & + \sum_{j=1}^m D_1(i,j) \left[\int_0^{u-b_{k-1}} V_{j,k}^n(u-x;B) dF_{i,j}(x) + \sum_{l=1}^{k-1} \int_{u-b_l}^{u-b_{l-1}} V_{j,l}^n(u-x;B) dF_{i,j}(x) \right] \\ = \; & 0. \end{aligned} \tag{5.32}$$

We can also rewrite (5.32) into matrix form as

$$\begin{aligned} c_k[\vec{V}_k^n(u;B)]' &= n\delta \vec{V}_k(u;B) - \boldsymbol{D_0} \vec{V}_k^n(u;B) - \int_0^{u-b_{k-1}} \boldsymbol{\Lambda_f}(x) \vec{V}_k^n(u-x;B) dx \\ & \quad -\vec{\zeta}_k^n(u), \quad b_{k-1} < u < b_k, \end{aligned} \tag{5.33}$$

where $\vec{V}_k^n(u;B) = (V_{1,k}^n(u;B), \cdots, V_{m,k}^n(u;B))^\top$ and $\vec{\zeta}_k^n(u) = (\zeta_{1,k}^n(u), \cdots, \zeta_{m,k}^n(u))^\top$ is a vector with the ith element in the following form for $b_{k-1} < u < b_k$:

$$\zeta_{i,k}^n(u) = n(c-c_k) V_{i,k}^{n-1}(u;B) + \sum_{j=1}^m D_1(i,j) \sum_{l=1}^{k-1} \int_{u-b_l}^{u-b_{l-1}} V_{j,l}^n(u-x;B) dF_{i,j}(x). \tag{5.34}$$

Note that this non-homogeneous term for the nth moment of the discounted dividend payments in the kth layer involves all the nth moment of the discounted dividend payments in the lower layers as well as the $(n-1)$th moment of the discounted dividend payments in the kth layer.

When $k = 1$, no previous layer is needed to be considered and $c_1 = c$. Thus the second term in (5.34) does not exist and the first term is equal to 0. Equation (5.33) is simplified to a homogeneous integro-differential equation and its solution is given by

$$\vec{V}_1^n(u;B) = \mathbf{v}_1^n(u) \vec{V}_1^n(0;B), \quad 0 \leq u < b_1.$$

When $k = 2, \cdots, n+1$, (5.33) satisfies a non-homogeneous integro-differential equation described in Lemma 2 with restriction $b_{k-1} \leq u < b_k$. The solution to (5.33) is given by

$$\vec{V}_k^n(u;B) = \mathbf{v}_k^n(u-b_{k-1}) \vec{V}_k^n(b_{k-1};B) - \frac{1}{c_k} \int_0^{u-b_{k-1}} \mathbf{v}_k^n(t) \vec{\zeta}_k^n(u-t) dt, \quad b_{k-1} \leq u < b_k, \tag{5.35}$$

where $\mathbf{v}_k^n(u)$ is given by a Laplace inversion of the inverse of a matrix similar to (3.11) with c and δ being replaced by c_k and $n\delta$. That is,

$$\mathbf{v}_k^n(u) = \mathcal{L}_s^{-1}\left\{\left[\left(s - \frac{n\delta}{c_k}\right)\boldsymbol{I} + \frac{1}{c_k}\left(\boldsymbol{D_0} + \boldsymbol{\Lambda}_{\hat{f}}(s)\right)\right]^{-1}\right\}. \tag{5.36}$$

The continuity condition for $\vec{V}_k^n(u;B)$ is

$$\vec{V}_k^n(b_k-;B) = \vec{V}_{k+1}^n(b_k+;B).$$

Similar to the case of the expected dividend payment in Section 5.1, the derivative of $\vec{V}_k^n(u;B)$ with respect to u is not continuous at each threshold level. In fact, it follows from (5.33) and (5.34) that

$$c_k\frac{d\vec{V}_k^n(u;B)}{du}\Big|_{u=b_k-} = c_{k+1}\frac{d\vec{V}_{k+1}^n(u;B)}{du}\Big|_{u=b_k+} + n(c-c_{k+1})\vec{V}_{k+1}^{n-1}(b_k+;B). \tag{5.37}$$

In order to achieve a recursive algorithm to obtain an analytical expression of (5.35), we need to have a starting point at the beginning and a boundary condition in the last layer. Again, the moments of the dividend payments prior to ruin under a constant barrier strategy plays an important role here. To be specific, we define

$$M_i(u,y;b) = \mathbb{E}_i[e^{yD_{u,b}}|U(0)=u], \quad 0 \le u < b, i \in E,$$

and

$$V_i^n(u;b) = \mathbb{E}_i[D_{u,b}^n|U(0)=u], \quad 0 \le u < b, i \in E,$$

to be the moment generating function and the nth moment of the dividend payments prior to ruin in the MAP risk model under a constant barrier strategy with barrier b, respectively, which can be viewed as special cases for the multi-threshold MAP risk model with $b_1 = b$ and $c_k = 0$ for $k > 0$. Then the nth moment of the dividend payments prior to ruin satisfies a system of integro-differential equations as

$$c[\vec{V}_k^n(u;b)]' = n\delta\vec{V}_k^n(u;b) - \boldsymbol{D_0}\vec{V}_k^n(u;b) - \int_0^u \boldsymbol{\Lambda_f}(x)\vec{V}_k^n(u-x;b)dx, \quad 0 \le u < b.$$

The solution to the homogeneous integro-differential equation above can be expressed in matrix form as

$$\vec{V}^n(u;b) = \mathbf{v}^n(u)\vec{V}^n(0;b), \quad 0 \le u < b, \tag{5.38}$$

where $\mathbf{v}^n(u)$ is given by

$$\mathbf{v}^n(u) = \mathcal{L}_s^{-1}\left\{\left[\left(s - \frac{n\delta}{c}\right)\boldsymbol{I} + \frac{1}{c}\left(\boldsymbol{D_0} + \boldsymbol{\Lambda}_{\hat{f}}(s)\right)\right]^{-1}\right\}.$$

For the case $u \geq b$, $\vec{V}^n(u;b) = \vec{V}^n(u;b) + u - b$. The continuity condition for $\vec{V}^n(u;b)$ at b is

$$\vec{V}^n(b-;b) = \vec{V}^n(b+;b),$$

and following from (5.37), the boundary condition is

$$\frac{d\vec{V}^n(u;b)}{du}\bigg|_{u=b} = n\vec{V}^{n-1}(b;b), \quad 0 \leq u < b. \tag{5.39}$$

Then the value of $\vec{V}^n(0;b)$ in (5.38) can be obtained from the following matrix equation,

$$(\mathbf{v}^n)'(b)\vec{V}^n(0;b) = n\vec{V}^{n-1}(b;b).$$

Thus we have

$$\begin{aligned}\vec{V}^n(u;b) &= \mathbf{v}^n(u)\vec{V}^n(0;b)\\ &= n\mathbf{v}^n(u)[(\mathbf{v}^n)'(b)]^{-1}\vec{V}^{n-1}(b;b)\\ &= n!\mathbf{v}^n(u)[(\mathbf{v}^n)'(b)]^{-1}\mathbf{v}^{n-1}(b)[(\mathbf{v}^{n-1})'(b)]^{-1}\cdots\mathbf{v}^1(b)[(\mathbf{v}^1)'(b)]^{-1}\vec{1}, \quad 0 \leq u < b.\end{aligned} \tag{5.40}$$

The way of setting up a recursive algorithm for the nth moment of the discounted dividend payments prior to ruin is similar to the one for the expected discounted dividend payments in Section 5.3. Once we have all the moments of the discounted dividend payments prior to ruin for each layer, the piecewise defined vector function $\vec{V}^n(u;B) = \vec{V}_k^n(u;B)$ for $b_{k-1} \leq u < b_k$ can be obtained. Theoretically, the moment generating function of the discounted dividend payments prior to ruin can be calculated based on the nth moments, $n = 1, 2, \cdots$, and the distribution of the discounted dividend payments is determined.

Chapter 6

Layer-Based Recursive Approach

As we can see from the recursive expressions in Chapter 4 for the expected discounted penalty function and Chapter 5 for the expected discounted dividend payments in the multi-threshold MAP model, a computational disadvantage of the differential approach based on the integro-differential equations is the fact that the obtained recursions among different layers have to be solved with unknown constants, which can only be evaluated in the last layer. It makes this method computational infeasible when the number of layers is large. In this chapter, we borrow the idea in Albrecher and Hartinger (2007) for the classical risk model to set up an alternative layer-based recursive approach in the multi-threshold MAP risk model, in which the complete solution of the k-layer MAP model can be obtained from the complete solution of the $(k-1)$-layer MAP model with the lower $k-1$ layers coincidence and the classical one-layer MAP model.

6.1 Time Value of "Upper Exit"

Recalling the concept of stopping time in (3.21) and (3.22), we denote $\tau_k^*(u,a,b)$, $\tau_k^+(u,a,b)$ and $\tau_k^-(u,a,b)$ to be three stopping times for the MAP risk model with k layers. Referring to Figure 1.1 in Appendix A, the structure of the surplus process implies that the upper exit of any interval $[a,b)$ can happen only through the continuous premium income, while the lower drop is due to the claim arrival. The time of ruin, which is originally defined in (1.2), is denoted as $\tau_k(u)$ in the k-layer risk model given the initial surplus u. Note that $\tau_k(u)$ is consistent with $\tau_k^-(u,0,\infty)$.

Let $\mathbf{1}_{[A]}$ be an indicator function on set A. Denote

$$B_{i,j,k}(u,b) = \mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b)}\mathbf{1}_{[J(\tau_k^+(u,0,b))=j]}\big|J(0)=i\right] \tag{6.1}$$

to be the Laplace transform of $\tau_k^+(u,0,b)$, given that the surplus process reaches b in MAP phase j and the initial phase is i; $B_{i,j,k}(u,b)$ can also be interpreted as the expected present

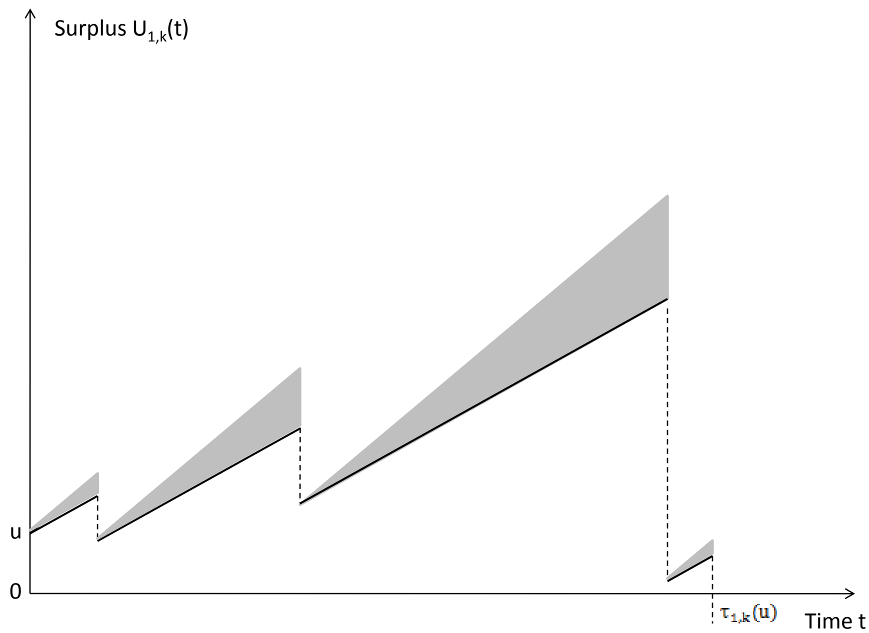


Figure 6.1: Sample path of the classical (one-layer) surplus process with dividend strategy

value of a payment of 1 at the time when the surplus reaches the level b for the first time provided that ruin has not occurred yet. Further let $\mathbf{B}_k(u,b)$ be a matrix with the (i,j)th element being $B_{i,j,k}(u,b)$. Note that the notation $\mathbf{B}_{k-1}(u,b)$ corresponds to $\mathbf{B}_k(u,b)$ with the top threshold b_{k-1} shifted to infinity. For example, in a three-threshold (four layers) MAP risk model we denote three thresholds as b_1, b_2, and b_3, where $0 < b_1 < b_2 < b_3 < \infty$. $\mathbf{B}_4(u,b)$ is the Laplace transform of the stopping time $\tau_4^+(u,0,b)$ in matrix form. When the top threshold b_3 is shifted to infinity, it becomes a two-threshold (three layers) MAP risk model and $\mathbf{B}_4(u,b)$ reduces to $\mathbf{B}_3(u,b)$.

Now we revisit the classical (one-layer) model and introduce a new notation $U_{1,k}(t)$, which denotes a surplus process at time t in the classical model with parameter c_k. Figure 6.1 illustrates a sample path of $U_{1,k}(t)$. In general, the subscript $\{\cdot\}_{1,k}$ refers to the corresponding quantity in the classical model with parameters c_k. For example, $\tau_{1,k}(u)$ is the time of ruin in the one-layer MAP risk model with premium rate c_k. Note that this model can be seen as a model with premium rate c and paying dividends at a rate $c - c_k$. Then the discounted dividend payments prior to ruin of the surplus process $U_{1,k}(t)$, denoted by $D_{1,k}(u)$, is defined as

$$D_{1,k}(u) = (c - c_k)\int_0^{\tau_{1,k}(u)} e^{-\delta t}dt.$$

The corresponding expected discounted dividend payments given the initial MAP phase i

and initial surplus u is

$$V_{i,(1,k)}(u) = \mathbb{E}_i\left[D_{1,k}(u)|U_{1,k}(0)=u\right], \quad i \in E.$$

Similar to Lemma 3.2 in Albrecher and Hartinger (2007), we have the following lemma stating the properties of $\mathbf{B}_k(u,b)$.

Lemma 3 *For $\delta > 0$ and $k \in \mathbb{N}^+$, we have*

1.

$$\begin{aligned} \mathbf{B}_k &= \mathbf{1}, \quad \text{if } u \geq b, \\ \mathbf{B}_k &= \mathbf{0}, \quad \text{if } u < 0; \end{aligned}$$

2. for $0 \leq u < b_{k-1}$,

$$\mathbf{B}_k(u,b) = \begin{cases} \mathbf{B}_{k-1}(u,b), & \text{if } b \leq b_{k-1}, \\ \mathbf{B}_{k-1}(u,b_{k-1})\mathbf{B}_k(b_{k-1},b), & \text{if } b \geq b_{k-1}; \end{cases}$$

3. for $b_{k-1} \leq u \leq b$,

$$\begin{aligned} \mathbf{B}_k(u,b) &= \mathbf{B}_{1,k}(u-b_{k-1}, b-b_{k-1}) + \mathbf{M}_k(u-b_{k-1}) \\ &\quad - \mathbf{B}_{1,k}(u-b_{k-1}, b-b_{k-1})\mathbf{M}_k(b-b_{k-1}), \end{aligned}$$

where $\mathbf{M}_k(v)$ is a matrix with the (i,j)th element being $M_{i,j,k}(v)$ given by

$$M_{i,j,k}(v) = \mathbb{E}\left[\sum_{l=1}^{m} e^{-\delta\tau_{1,k}(v)} B_{l,j,k}(b_{k-1} - |U_{1,k}(\tau_{1,k}(v))|, b)\mathbf{1}_{[J(\tau_{1,k}(v))=l]}\big|J(0)=i\right].$$

Proof. In Case *1*, when the initial surplus is greater than or equal to b, no time is needed for the surplus process to reach b; that is, $\tau_k^+(u,0,b) = 0$. When the initial surplus is less than 0, ruin has occurred. The time of the surplus process to reach $b > 0$ is infinity. So Case *1* is true.

In Case *2*, the surplus process with k layers coincides with the process with $k-1$ layers before the first exit of the interval $[0, b_{k-1})$, we have $\tau_k^+(u,0,b) = \tau_{k-1}^+(u,0,b)$ and then $B_{i,j,k}(u,b) = B_{i,j,k-1}(u,b)$ for $0 \leq b \leq b_{k-1}$. Putting it in matrix form, for $0 \leq b \leq b_{k-1}$, we have

$$\mathbf{B}_k(u,b) = \mathbf{B}_{k-1}(u,b).$$

For $b \geq b_{k-1}$, the surplus process of upcrossing b is equal to the surplus process which upcrosses b_{k-1} first and then continues to upcross b with a new initial surplus b_{k-1}. Thus, for $b \geq b_{k-1}$, we have

$$\begin{aligned} B_{i,j,k}(u,b) &= \mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b)}\mathbf{1}_{[J(\tau_k^+(u,0,b))=j]}\Big|J(0)=i\right] \\ &= \mathbb{E}\left[e^{-\delta[\tau_{k-1}^+(u,0,b_{k-1})+\tau_k^+(b_{k-1},0,b)]}\mathbf{1}_{[J(\tau_k^+(u,0,b))=j]}\Big|J(0)=i\right] \\ &= \sum_{l=1}^{m}\left\{\mathbb{E}\left[e^{-\delta\tau_{k-1}^+(u,0,b_{k-1})}\mathbf{1}_{[J(\tau_{k-1}^+(u,0,b_{k-1}))=l]}\Big|J(0)=i\right]\right. \\ &\qquad \left.\mathbb{E}\left[e^{-\delta\tau_k^+(b_{k-1},0,b)}\mathbf{1}_{[J(\tau_k^+(b_{k-1},0,b))=j]}\Big|J(\tau_{k-1}^+(u,0,b_{k-1}))=l\right]\right\} \\ &= \sum_{l=1}^{m} B_{i,l,k-1}(u,b_{k-1})B_{l,j,k}(b_{k-1},b), \quad 0 \leq u < b_{k-1}. \end{aligned}$$

Putting it in a matrix form, for $b \geq b_{k-1}$, we have

$$\mathbf{B}_k(u,b) = \mathbf{B}_{k-1}(u,b_{k-1})\mathbf{B}_k(b_{k-1},b), \quad 0 \leq u < b_{k-1}. \tag{6.2}$$

In Case *3*, when $b_{k-1} \leq u \leq b$, two sample surplus paths to reach level b are illustrated in Figure A.4 and A.5. The surplus process arrives at b directly without dropping to b_{k-1} in Figure A.4. In this case, $\tau_k^+(u,0,b)$ is equal to $\tau_k^+(u,b_{k-1},b)$. Figure A.5 describes the case that the surplus process drops below level b_{k-1} first without causing ruin and then increase to level b from a new initial surplus. Thus, we have,

$$\begin{aligned} &\mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b)}\mathbf{1}_{[J(\tau_k^+(u,0,b))=j]}\Big|J(0)=i\right] \\ =\ &\mathbb{E}\left[e^{-\delta\tau_k^+(u,b_{k-1},b)}\mathbf{1}_{[U_B(\tau_k^*(u,b_{k-1},b))=b,J(\tau_k^*(u,b_{k-1},b))=j]}\Big|J(0)=i\right] \\ &+\mathbb{E}\left[e^{-\delta[\tau_k^-(u,b_{k-1},b)+\tau_k^+(U_B(\tau_k^-(u,b_{k-1},b)),0,b)]}\mathbf{1}_{[U_B(\tau_k^*(u,b_{k-1},b))<b,J(\tau_k^*(u,b_{k-1},b))=j]}\Big|J(0)=i\right] \\ =\ &P_{k,1}(u)+P_{k,2}(u), \quad b_{k-1} \leq u \leq b. \end{aligned} \tag{6.3}$$

In order to link equation (6.3) to $B_{i,j,k}$ defined in (6.1), we resort to the surplus process $U_{1,k}(t)$ under the classical one-layer model. Three sample surplus paths of this surplus process are illustrated in Figure A.6, A.7 and A.8. Figure A.6 describes the case that the surplus process starts from $u - b_{k-1}$ and reaches $b - b_{k-1}$ prior to ruin. When the premium rate of this surplus process is the same as the one of the kth (top) layer from a $(k-1)$-threshold MAP risk model, the time of upcrossing level $b - b_{k-1}$ at phase j prior to ruin from initial surplus $u - b_{k-1}$ in Figure A.6 is equal to the time of upcrossing level b at phase j without dropping to b_{k-1} from initial surplus u in Figure A.4, both given the initial phase i. That is, $\tau_k^+(u,b_{k-1},b) = \tau_{1,k}^+(u-b_{k-1},0,b-b_{k-1})$. We can rewrite the first term in equation (6.3) as

$$\begin{aligned} P_{k,1}(u) &= \mathbb{E}\left[e^{-\delta\tau_{1,k}^+(u-b_{k-1},0,b-b_{k-1})}\mathbf{1}_{[J(\tau_{1,k}^+(u-b_{k-1},0,b-b_{k-1}))=j]}\Big|J(0)=i\right] \\ &= B_{i,j,(1,k)}(u-b_{k-1},b-b_{k-1}), \quad b_{k-1} \leq u \leq b. \end{aligned}$$

Figure A.7 describes the case that ruin occurs before the surplus process reaches $b-b_{k-1}$. The dash line indicates that it recovers from ruin, if possible, and further reaches $b-b_{k-1}$. Similarly, the stopping time $\tau_k^-(u, b_{k-1}, b)$ in Figure A.5 corresponds to the stopping time $\tau_{1,k}^-(u-b_{k-1}, 0, b-b_{k-1})$ in the surplus process $U_{1,k}(t)$. We follow the idea in the proof of Lemma 3.2 in Albrecher and Hartinger (2007) to replace $\tau_k^-(u, b_{k-1}, b)$ by the time of ruin $\tau_{1,k}(u-b_{k-1})$. When ruin happens in the surplus process $U_{1,k}(t)$, the deficit of ruin is $|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|$. It corresponds to the case of the surplus process U_B (t) dropping below b_{k-1} and start again with an initial surplus $b_{k-1}-|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|$. However, the event of ruin in the classical model does not distinguish whether the surplus process hits $b-b_{k-1}$ before or not prior to ruin. If $U_{1,k}(t)$ reaches $b-b_{k-1}$ before ruin, as illustrated in Figure A.8, it is the same case as $U_B(t)$ reaches b before dropping to b_{k-1}, and is already considered in the first term of equation (6.3). Those trajectories need to be corrected. Taking the interim transition states into account, we have the following equation for the second term in equation (6.3)

$$\begin{aligned} & P_{k,2}(u) \\ = \; & \mathbb{E}\left[\sum_{l=1}^{m} e^{-\delta\tau_{1,k}(u-b_{k-1})} B_{l,j,k}(b_{k-1}-|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|, b)\mathbf{1}_{[J(\tau_{1,k}(v))=l]}\big| J(0)=i\right] \\ & -\mathbb{E}\left[\sum_{l'=1}^{m}\sum_{l=1}^{m} B_{i,l',(1,k)}(u-b_{k-1}, b-b_{k-1})e^{-\delta\tau_{1,k}(b-b_{k-1})}\right. \\ & \left. B_{l,j,k}(b_{k-1}-|U_{1,k}(\tau_{1,k}(b-b_{k-1}))|, b)\mathbf{1}_{[J(\tau_{1,k}(v))=l]}\big| J\left(\tau_{1,k}^{+}(u-b_{k-1}, 0, b-b_{k-1})\right)=l'\right]. \end{aligned} \tag{6.4}$$

Let

$$M_{i,j,k}(v) = \mathbb{E}\left[\sum_{l=1}^{m} e^{-\delta\tau_{1,k}(v)} B_{l,j,k}(b_{k-1}-|U_{1,k}(\tau_{1,k}(v))|, b)\mathbf{1}_{[J(\tau_{1,k}(v))=l]}\big| J(0)=i\right],$$

and $\mathbf{M}_k(v)$ be a matrix with the (i,j)th element being $M_{i,j,k}(v)$. Then $P_{k,2}$ can be rewritten as

$$P_{k,2}(u) = M_{i,j,k}(u-b_{k-1}) - \sum_{l'=1}^{m} B_{i,l',(1,k)}(u-b_{k-1}, b-b_{k-1})M_{l',j,k}(b-b_{k-1}).$$

Further combining $P_{k,1}$ and $P_{k,2}$, we have

$$\begin{aligned} \mathbf{B}_k(u,b) \;\; = \;\; & \mathbf{B}_{1,k}(u-b_{k-1}, b-b_{k-1}) + \mathbf{M}_k(u-b_{k-1}) \\ & -\mathbf{B}_{1,k}(u-b_{k-1}, b-b_{k-1})\mathbf{M}_k(b-b_{k-1}), \quad b_{k-1} \le u \le b. \end{aligned}$$

□

Let $\mathbf{A}_{1,k}(v)$ be a matrix with the (i,j)th element $A_{i,j,(1,k)}(v)$ being a random variable such that

$$\Pr[A_{i,j,(1,k)}(v) = a] = \Pr[e^{-\delta\tau_{1,k}(v)}\mathbf{1}_{[J(\tau_{1,k}(v))=j]} = a\big|J(0) = i].$$

Then $\mathbf{M}_k(v)$ can be expressed as

$$\mathbf{M}_k(v) = \mathbb{E}\left[\mathbf{A}_{1,k}(v)\mathbf{B}_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(v))|, b)\right]. \tag{6.5}$$

Let $\mathbf{M}^*_k(v)$ be a matrix with the (i,j)th element being $M^*_{i,j,k}(v)$ given by

$$M^*_{i,j,k}(v) = \mathbb{E}\left[\sum_{l=1}^{m} e^{-\delta\tau_{1,k}(v)} B_{l,j,k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(v))|, b_{k-1})\mathbf{1}_{[J(\tau_{1,k}(v))=l]}\big|J(0) = i\right].$$

Similar to $\mathbf{M}_k(v)$, with the help of matrix $\mathbf{A}_{1,k}(v)$, we have an alternative expression for $\mathbf{M}^*_k(v)$ as

$$\mathbf{M}^*_k(v) = \mathbb{E}\left[\mathbf{A}_{1,k}(v)\mathbf{B}_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(v))|, b_{k-1})\right]. \tag{6.6}$$

Now we obtain, from Case *2* of Lemma 3 at $u = 0$, that

$$\mathbf{B}_k(0, b) = \mathbf{B}_{k-1}(0, b_{k-1})\mathbf{B}_k(b_{k-1}, b), \tag{6.7}$$

and from Case *3* of Lemma 3 at $u = b_{k-1}$, that

$$\mathbf{B}_k(b_{k-1}, b) = \mathbf{B}_{1,k}(0, b - b_{k-1}) + \mathbf{M}_k(0) - \mathbf{B}_{1,k}(0, b - b_{k-1})\mathbf{M}_k(b - b_{k-1}). \tag{6.8}$$

The relationship between $\mathbf{M}_k(v)$ and $\mathbf{M}^*_k(v)$ is then found by substituting (6.2) and (6.7) into (6.5) and (6.6), respectively, that

$$\mathbf{M}_k(v) = \mathbf{M}^*_k(v)\mathbf{B}^{-1}_{k-1}(0, b_{k-1})\mathbf{B}_k(0, b). \tag{6.9}$$

Combining (6.2), (6.8) and (6.9), we obtain

$$\begin{aligned}
\mathbf{B}_k(b_{k-1}, b) &= \mathbf{B}_{k-1}(b_{k-1}, b_{k-1})\mathbf{B}_k(b_{k-1}, b) \\
&= \mathbf{B}_{1,k}(0, b - b_{k-1}) + \mathbf{M}^*_k(0)\mathbf{B}^{-1}_{k-1}(0, b_{k-1})\mathbf{B}_k(0, b) \\
&\quad -\mathbf{B}_{1,k}(0, b - b_{k-1})\mathbf{M}^*_k(b - b_{k-1})\mathbf{B}^{-1}_{k-1}(0, b_{k-1})\mathbf{B}_k(0, b).
\end{aligned}$$

After some manipulations, $\mathbf{B}_k(0, b)$ can be obtained from the equation above. Note that $\mathbf{B}_k(0, b)$ solely depends on quantities from the classical one-layer model with parameters from the kth layer in the multi-layer risk model such as $\tau_{1,k}(u)$ and $\mathbf{B}_{1,k}(u, b)$, and quantities from the lower layer such as $\mathbf{B}^{-1}_{k-1}(u, b)$.

6.2 The Expected Discounted Dividends

In this section, we show how to use the quantities of Section 6.1 to calculate the expected discounted dividend payments in the MAP risk model with k layers.

For $0 \le u < b_{k-1}$, conditioning on the event of either reaching b_{k-1} first or ruin occurring without reaching b_{k-1}, we have

$$\begin{aligned} & V_{i,k}(u;B) \\ = & \mathbb{E}\left[\int_0^{\tau_k(u)} e^{-\delta t} dD_{u,B}(t) | J(0)=i\right] \\ = & \mathbb{E}\left[\int_0^{\tau_k^*(u,0,b_{k-1})} e^{-\delta t} dD_{u,B}(t) | J(0)=i\right] + \mathbb{E}\left[\int_{\tau_k^*(u,0,b_{k-1})}^{\tau_k(u)} e^{-\delta t} dD_{u,B}(t) | J(0)=i\right] \\ = & I_{k,1}(u) + I_{k,2}(u). \end{aligned} \tag{6.10}$$

If ruin occurs before the surplus process reaches b_{k-1}, $\tau_k^*(u,0,b_{k-1}) = \tau_k^-(u,0,b_{k-1}) = \tau_k(u)$. The second term in the equation above goes to 0. If this is the case, the dividend payments in the k-layer model is the same as the one in the $(k-1)$-layer model. That is, $V_{i,k}(u;B) = V_{i,k-1}(u;B)$. Otherwise, the surplus process hits b_{k-1} prior to ruin, and $\tau_k^*(u,0,b_{k-1}) = \tau_k^+(u,0,b_{k-1}) = \tau_{k-1}^+(u,0,b_{k-1})$. After it hits b_{k-1} in phase l, $l \in E$, the surplus process begins with a new initial surplus b_{k-1}. The expected discounted dividend payments thereafter are calculated as $V_{l,k}(b_{k-1};B)$, and then we discount it back to time 0. It follows that

$$\begin{aligned} I_{k,2} &= \sum_{l=1}^{m} \mathbb{E}\left[e^{-\delta\tau_{k-1}^+(u,0,b_{k-1})} \mathbf{1}_{[J(\tau_{k-1}^+(u,0,b_{k-1}))=l]} \big| J(0)=i\right] V_{l,k}(b_{k-1};B) \\ &= \sum_{l=1}^{m} B_{i,l,k-1}(u,b_{k-1}) V_{l,k}(b_{k-1};B). \end{aligned}$$

The first term of (6.10) corresponds to the expected discounted dividend payments of the surplus process hitting b_{k-1} before ruin and it can be calculated as

$$\begin{aligned} I_{k,1} &= \mathbb{E}\left[\int_0^{\tau_k^*(u,0,b_{k-1})} e^{-\delta t} dD_{u,B}(t) \big| J(0)=i\right] \\ &= \mathbb{E}\left[\int_0^{\tau_{k-1}^+(u,0,b_{k-1})} e^{-\delta t} dD_{u,B}(t) \big| J(0)=i\right] \\ &= \mathbb{E}\left[\int_0^{\tau_{k-1}(u)} e^{-\delta t} dD_{u,B}(t) \big| J(0)=i\right] \\ &\quad - \sum_{l=1}^{m} \Bigg\{ \mathbb{E}\left[e^{-\delta \tau_{k-1}^+(u,0,b_{k-1})} \mathbf{1}_{[J(\tau_{k-1}^+(u,0,b_{k-1}))=l]} \big| J(0)=i\right] \\ &\qquad \mathbb{E}\left[\int_{\tau_{k-1}^+(u,0,b_{k-1})}^{\tau_{k-1}(b_{k-1})} e^{-\delta t} dD_{u,B}(t) \big| J(\tau_{k-1}^+(u,0,b_{k-1}))=l\right] \Bigg\} \\ &= V_{i,k-1}(u;B) - \sum_{l=1}^{m} B_{i,j,k-1}(u,b_{k-1}) V_{l,k-1}(b_{k-1};B). \end{aligned}$$

The third equality in the equation above holds because the discounted dividend payments in the $(k-1)$-layer model prior to ruin can be divided into two parts. One is the discounted dividend payments prior to the time of upcrossing level b_{k-1}, which is the one we are interested in. The other is the discounted dividend payments accumulated from the time of starting at b_{k-1} to the time of ruin in the $(k-1)$-layer model, which is the one needed to be subtracted from the total discounted dividend payments prior to ruin. We can evaluate the second part by first discounting the dividend payments gained after hitting b_{k-1} to the time of hitting b_{k-1} and further discounting it to time 0. Hence, equation (6.10) can be rewritten as

$$V_{i,k}(u;B) = V_{i,k-1}(u;B) + \sum_{l=1}^{m} B_{i,l,k}(u,b_{k-1}) \left[V_{l,k}(b_{k-1};B) - V_{l,k-1}(b_{k-1};B)\right], \qquad (6.11)$$

and in matrix form,

$$\vec{V}_k(u;B) = \vec{V}_{k-1}(u;B) + \mathbf{B}_{k-1}(u,b_{k-1}) \left[\vec{V}_k(b_{k-1};B) - \vec{V}_{k-1}(b_{k-1};B)\right]. \qquad (6.12)$$

When $u=0$ in (6.11) and (6.12), we have

$$V_{i,k}(0;B) = V_{i,k-1}(0;B) + \sum_{l=1}^{m} B_{i,l,k}(0,b_{k-1}) \left[V_{l,k}(b_{k-1};B) - V_{l,k-1}(b_{k-1};B)\right], \qquad (6.13)$$

and

$$\vec{V}_k(0;B) = \vec{V}_{k-1}(0;B) + \mathbf{B}_{k-1}(0,b_{k-1}) \left[\vec{V}_k(b_{k-1};B) - \vec{V}_{k-1}(b_{k-1};B)\right].$$

Thus we arrive, provided that the $\mathbf{B}_{k-1}^{-1}(0,b_{k-1})$ is invertible, at

$$\vec{V}_k(u;B) = \vec{V}_{k-1}(u;B) + \mathbf{B}_{k-1}(u,b_{k-1}) \mathbf{B}_{k-1}^{-1}(0,b_{k-1}) \left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right]. \qquad (6.14)$$

For $u \geq b_{k-1}$, we employ the techniques used in deriving Lemma 3 to link the quantities in the k-layer MAP risk model to the one-layer model with parameter c_k. Conditioning on the events of dropping below b_{k-1} or not, we have

$$\begin{aligned} & V_{i,k}(u;B) \\ = \; & V_{i,(1,k)}(u-b_{k-1}) + \mathbb{E}\left[\sum_{l=1}^{m} e^{-\delta\tau_{1,k}(u-b_{k-1})} V_{l,k}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|;B) \right. \\ & \left. \mathbf{1}_{[J(\tau_{1,k}(u-b_{k-1}))=l]} \middle| J(0)=i \right]. \end{aligned}$$

Putting it in matrix form and using (6.14), we obtain

$$\begin{aligned} & \vec{V}_k(u;B) \\ = \; & \vec{V}_{1,k}(u-b_{k-1}) + \mathbb{E}\left[\mathbf{A}_{1,k}(u-b_{k-1})\vec{V}_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|;B)\right] \\ = \; & \vec{V}_{1,k}(u-b_{k-1}) + \mathbb{E}\left[\mathbf{A}_{1,k}(u-b_{k-1})\left\{\vec{V}_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|;B)\right.\right. \\ & \left.\left. +\mathbf{B}_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|, b_{k-1})\mathbf{B}_{k-1}^{-1}(0,b_{k-1})\left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right]\right\}\right] \\ = \; & \vec{V}_{1,k}(u-b_{k-1}) + \mathbb{E}\left[\mathbf{A}_{1,k}(u-b_{k-1})\vec{V}_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|;B)\right] \\ & +\mathbf{M}_k^*(u-b_{k-1})\mathbf{B}_{k-1}^{-1}(0,b_{k-1})\left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right]. \end{aligned} \tag{6.15}$$

According to the continuity of $\vec{V}_k(u;B)$ at $u = b_{k-1}$, we evaluate (6.14) and (6.15) at $u = b_{k-1}$ to obtain

$$\begin{aligned} & \vec{V}_k(b_{k-1};B) \\ = \; & \vec{V}_{k-1}(b_{k-1};B) + \mathbf{B}_{k-1}(b_{k-1},b_{k-1})\mathbf{B}_{k-1}^{-1}(0,b_{k-1})\left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right] \\ = \; & \vec{V}_{1,k}(0) + \mathbb{E}\left[\mathbf{A}_{1,k}(0)\vec{V}_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|;B)\right] \\ & +\mathbf{M}_k^*(0)\mathbf{B}_{k-1}^{-1}(0,b_{k-1})\left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right], \end{aligned}$$

and further get

$$\begin{aligned} & \mathbf{B}_{k-1}^{-1}(0,b_{k-1})\left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right] \\ = \; & [\mathbf{I} - \mathbf{M}_k^*(0)]^{-1}\left\{\vec{V}_{1,k}(0) - \vec{V}_{k-1}(b_{k-1};B)\mathbb{E}\left[\mathbf{A}_{1,k}(0)\vec{V}_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|;B)\right]\right\}. \end{aligned} \tag{6.16}$$

Subsequently we can evaluate $\vec{V}_k(u;B)$ in (6.14) and (6.15) by replacing $\mathbf{B}_{k-1}^{-1}(0,b_{k-1})\left[\vec{V}_k(0;B) - \vec{V}_{k-1}(0;B)\right]$ with (6.16).

Note that in (6.14), (6.15) and (6.16), for all values of u, $\vec{V}_k(u;B)$ is expressed as a function of $\vec{V}_{k-1}(u;B)$ and $\vec{V}_k(u;b)$, which is related to $\tau_{1,k}(u)$. That implies that we can recursively determine the solution beginning from the classical one-layer model and adding one more layers once a time based on the previous coincidence layers.

Chapter 7

Numerical Examples

In this chapter, we illustrate the applicability of the differential recursive approach derived in Chapter 4 for the ruin probability as a special case of the expected discounted penalty function and in Chapter 5 for the expected discounted dividend payments prior to ruin. First, we consider the multi-threshold classical compound Poisson risk model with a two-threshold dividend strategy as the simplest case. Then we consider the case where the inter-claim times follow an Erlang(2) distribution as a special case of the Sparre-Andersen risk model. Further we give examples where the claim arrival process follows a Markov-modulated Poisson process with two phases, which may reflect the external environment effects due to normal or abnormal risk. The final example illustrates the case where the claim amounts are related to the transition changes. In all of the examples, we assume that three layers (two thresholds) are involved and the claim sizes are exponentially distributed.

Example 1 (*Probability of Ruin in the Classical Compound Poisson Risk Model*)

As a special case of the MAP risk model, the classical compound Poisson risk model has only one phase; that is, $m = 1$. In this case, we have $\boldsymbol{D_0} = -\lambda$ and $\boldsymbol{D_1} = \lambda$, where λ is the parameter in a Poisson process modeling the number of claims up to time t, $N(t)$. Further setting discounting factor $\delta = 0$ and penalty function $w(x, y) = 1$, (4.1) reduces to the probability of ruin in the multi-threshold MAP risk model. That is a piecewise function as

$$\varphi(u; B) = \begin{cases} \varphi_1(u; B) & 0 \leq u < b_1, \\ \varphi_2(u; B) & b_1 \leq u < b_2, \\ \varphi_3(u; B) & b_2 \leq u < \infty, \end{cases}$$

and for $k = 1, 2, 3$,

$$\varphi_k(u; B) = \mathbb{E}[I(\tau < \infty)|U(0) = u] = \Pr[\tau < \infty|U(0) = u], \quad b_{k-1} \leq u < b_k.$$

When the claim amounts are exponentially distributed with mean $1/\mu$ with Laplace transform $\mu/(s+\mu)$, $\boldsymbol{L}_c(s)$ in equation (3.9) reduces to $L_c(s)$ in the form of

$$L_c(s) = s - \frac{\lambda}{c} + \frac{\lambda\mu}{c(s+\mu)}.$$

Let θ be the security loading factor such that

$$c = \frac{(1+\theta)\lambda}{\mu}.$$

The solutions of equation $L_c(s) = 0$ are 0 and $-\theta\mu/(1+\theta) < 0$. It is discussed in Lin and Pavlova (2006) that the probability of ruin, $\varphi(u)$, under the classical compound Poisson model with premium rate c has the explicit form

$$\varphi(u) = \frac{1}{1+\theta} e^{-Ru},$$

where $R = \theta\mu/(1+\theta)$ is the so-called adjustment coefficient of Lundberg equation. Furthermore, the Laplace inversion of the inverse of $L_c(s)$, which is $\mathbf{v}(u)$ given by equation (3.11), is obtained explicitly as

$$\mathbf{v}(u) = 1 + \frac{1 - e^{-Ru}}{\theta} = \frac{1+\theta}{\theta}[1 - \varphi(u)].$$

Let $\varphi_1(u)$ be the probability of ruin under the classical compound Poisson risk model with premium rate c_1 and θ_1 be the security loading factor in the first layer. We have the following expression for the first layer,

$$\varphi_1(u; B) = \varphi_1(u) + \frac{\kappa_1(1+\theta_1)}{\theta_1}[1 - \varphi_1(u)], \quad 0 \le u < b_1.$$

Details of the derivation of the probability of ruin for the higher layers can be found in Lin and Sendova (2008).

We follow the parameters in Albrecher and Hartinger (2007) that $\lambda = 1$, $\beta = 1$, $b_1 = 5$, $b_2 = 10$, $c_1 = 1.4$, $c_2 = 1.3$ and $c_3 = 1.2$, and consider two threshold sets, $B_1 = \{0, b_1, \infty\}$ and $B_2 = \{0, b_1, b_2, \infty\}$. Then we have explicit formulas for different dividend strategies as follows:

- no dividend strategy,
$$\varphi(u) = 0.7143e^{-0.2857u}, \quad u \ge 0;$$

- one-threshold dividend strategy,
$$\varphi(u; B_1) = \begin{cases} 0.6757e^{-0.2857u} + 0.0540, & 0 \le u < 5, \\ 0.6845e^{-0.2308u}, & u \ge 5; \end{cases}$$

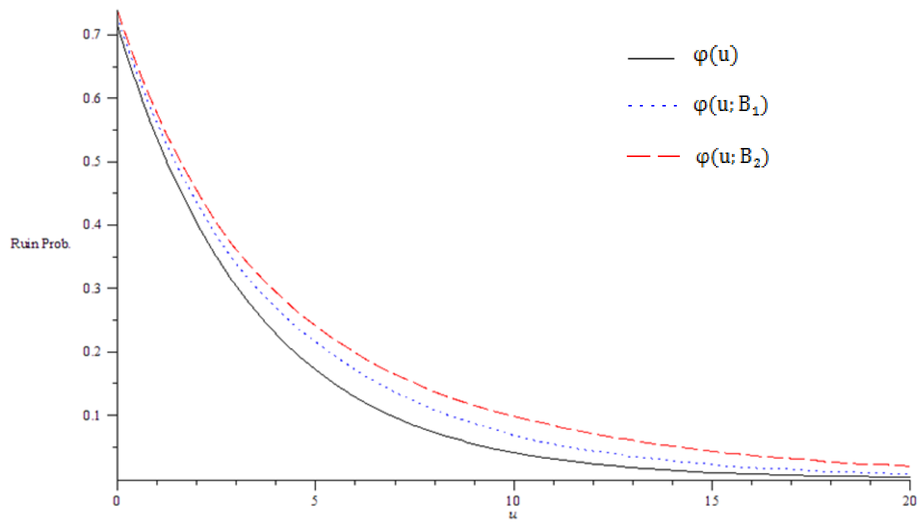


Figure 7.1: Ruin probabilities in the classical compound Poisson risk model under different dividend strategies

- two-threshold dividend strategy,

$$\varphi(u; B_2) = \begin{cases} 0.6536e^{-0.2857u} + 0.84926, & 0 \le u < 5, \\ 0.6622e^{-0.2308u} + 0.03271, & 5 \le u < 10, \\ 0.5199e^{-0.1667u}, & u \ge 10. \end{cases}$$

Figure 7.1 depicts the ruin probabilities as a function of u for compound Poisson risk models with no dividend strategy, with a one-threshold strategy and with a two-threshold strategy. As expected, the ruin probabilities decrease as the initial surplus increases. The ruin probabilities in the model under a multi-threshold strategy always increases with the number of thresholds.

Example 2 (*Probability of Ruin in the Sparre Andersen Risk Model*)

In this example, we consider the distribution of inter-claim times to be Erlang (2) with parameter λ. That is, $\vec{a} = (1, 0)^\top$, $\vec{s} = (0, \lambda)^\top$, and

$$\boldsymbol{S} = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}.$$

In this case, the density function, cumulative function and the Laplace transform of the inter-claim times are given by,

$$\begin{aligned} f_Z(x) &= \lambda^2 x e^{-\lambda x}, \quad x \geq 0, \\ F_Z(x) &= 1 - e^{-\lambda x} - e^{-\lambda x}\lambda x, \quad x \geq 0, \\ \hat{f}_Z(s) &= \frac{\lambda^2}{(s+\lambda)^2}. \end{aligned}$$

As mentioned in Chapter 2, the Sparre-Andersen risk model is a special case of the MAP risk model with $\boldsymbol{D_0} = \boldsymbol{S}$ and $\boldsymbol{D_1} = -\boldsymbol{S}\vec{1}\vec{\alpha}^\top$. It is obtained from equation (3.9) that matrix $\boldsymbol{L}_c(s)$ has the following form:

$$\boldsymbol{L}_c(s) = \frac{1}{c}\begin{pmatrix} cs-\lambda & \lambda \\ \frac{\lambda\beta}{s+\beta} & cs-\lambda \end{pmatrix},$$

and it follows from Example 1 in Li (2008) and Example 1 in Lu and Li (2009b) that $\boldsymbol{H}$ is a transpose of a Vandermonde matrix, which is a matrix with geometric progression in each row; it is given by

$$\boldsymbol{H} = \frac{1}{c}\begin{pmatrix} 1 & 1 \\ \frac{\lambda-c\rho_1}{\lambda} & \frac{\lambda-c\rho_2}{\lambda} \end{pmatrix},$$

with ρ_1 and ρ_2 are solutions to equation

$$0 = \det[\boldsymbol{L}_c(s)] = \frac{1}{c}\left[(cs-\lambda)^2 - \frac{\lambda^2\beta}{s+\beta}\right] \tag{7.1}$$

in the right half complex plain.

When the claim amounts are exponentially distributed, it is derived in Gerber and Shiu (2005) that $\theta_i(u)$ given in (3.27) is of the form

$$\theta_i(u) = \frac{\beta - R}{\beta + \rho_i} e^{-Ru}, \quad i = 1, 2,$$

where the Lundberg adjustment coefficient R is the absolute value of the negative solution to equation (7.1). Now the expression of $\mathbf{v}_k(u)$ can be computed for $k = 1, 2, 3$.

We choose $\lambda = 2$ such that the expected number of claim arrivals up to time t, $\mathbb{E}[N(t)]$, equals to the one in Example 1. Other parameters such as premium rates, threshold levels and claim amount distribution remain the same as those in Example 1. Following the recursive algorithm described in Chapter 4, the ruin probabilities for the Sparre-Andersen risk model with Erlang (2) inter-claim times under different dividend strategies are given by

- no dividend strategy,

$$\varphi(u) = 0.6330 e^{-0.3670u}, \quad u \geq 0;$$

- one-threshold dividend strategy,

$$\varphi(u;B_1)=\begin{cases}-8.5041\times 10^{-9}e^{2.2242u}+0.03657+0.6098e^{-0.3670u}, & 0\le u<5,\\ 0.5941e^{-0.2989u}, & u\ge 5;\end{cases}$$

- two-threshold dividend strategy,

$$\varphi(u;B_2)=\begin{cases}-1.4591\times 10^{-7}e^{2.2242u}+0.0521+0.5998e^{-0.3670u}, & 0\le u<5,\\ 4.1927\times 10^{-14}e^{2.3758u}+0.0002+0.6141e^{-0.2989u}, & 5\le u<10,\\ 0.2825e^{-0.2178u}, & u\ge 10.\end{cases}$$

Figure 7.2 shows the ruin probabilities as a function of u for this model under no dividend strategy, a one-threshold strategy and a two-threshold strategy. We reach the same conclusions as in Example 1 that, the more initial surplus we have, the lower ruin probabilities we face. Also the ruin probabilities in the model under a multi-threshold strategy always increases with the number of thresholds. It is interesting to note that, with the same claim amount distribution and expected claim numbers, the ruin probabilities in the Sparre Andersen risk model are lower than those in the classical compound Poisson risk model under the same dividend strategy.

Example 3 (*Probability of Ruin in the Markov-Modulated Risk Model*)

In this example, we consider a two-state Markov-modulated risk model with a two-threshold dividend strategy, that is, $\{J(t);t\ge 0\}$ is assumed to be a two-state Markov process, which can be interpreted as the external environment effects due to normal risk or abnormal risk. Under the normal risk condition, the claim arrivals are assumed to follow a Poisson distribution with parameter λ_1. While under the abnormal risk condition, they follow a Poisson distribution with λ_2. The intensity of transition from state 1 to state 2 is α_{12} and from state 2 to state 1 is α_{21}. Further the claim amount distributions f_1 and f_2 are exponentially distributed with means $1/\beta_1$ and $1/\beta_2$. It is reasonable to assume that $\lambda_1>\lambda_2$ and $\beta_1<\beta_2$. That is, we would have more claims with larger claim amounts in one environment state and less claims with smaller claim amounts in the other state. As introduced in Chapter 2, in the Markov-modulated risk model case,

$$\boldsymbol{D_1}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\quad \boldsymbol{D_0}=\boldsymbol{\Lambda}-\boldsymbol{D_1}=\begin{pmatrix}-\alpha_{12}-\lambda_1 & \alpha_{12}\\ \alpha_{21} & -\alpha_{21}-\lambda_2\end{pmatrix},$$

and, matrix $\boldsymbol{L}_c(s)$ given by (3.9) has the form

$$\boldsymbol{L}_c(s)=\frac{1}{c}\begin{pmatrix}cs-\alpha_{12}-\lambda_1+\frac{\lambda_1\beta_1}{s+\beta_1} & \alpha_{12}\\ \alpha_{21} & cs-\alpha_{21}-\lambda_2+\frac{\lambda_2\beta_2}{s+\beta_2}\end{pmatrix}.$$

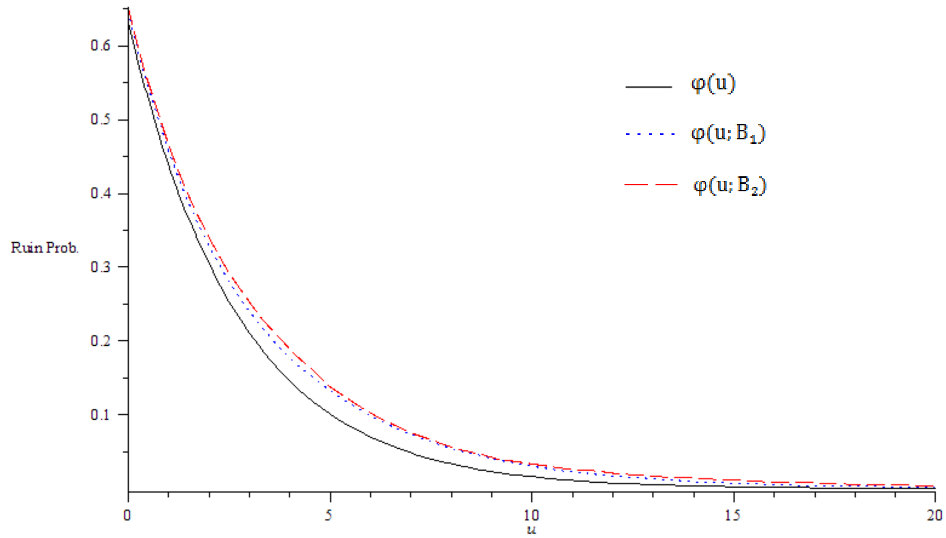


Figure 7.2: Ruin probabilities in the Sparre Andersen risk model with Erlang(2) inter-claim times under different dividend strategies

We denote ρ_l, $l = 1, 2, 3, 4$ to be the solutions of equation $\det[\boldsymbol{L}_c(s)] = 0$ and assume that they are distinct for simplicity. That is, we can obtain four distinct roots which satisfy the following equation,

$$0 = \frac{1}{c}\left[\left(cs - \alpha_{12} - \lambda_1 + \frac{\lambda_1\beta_1}{s+\beta_1}\right)\left(cs - \alpha_{21} - \lambda_2 + \frac{\lambda_2\beta_2}{s+\beta_2}\right) - \alpha_{12}\alpha_{21}\right]. \tag{7.2}$$

Recall that there are two approaches to evaluate $\mathbf{v}(u)$ in Section 3.3. In the case where the claim amounts are exponentially distributed, Li and Lu (2007) obtained an explicit expression for $\mathbf{v}(u) = [v_{i,j}(u)]_{i,j=1}^{m}$ based on the method of direct Laplace inversion as

$$v_{i,j}(u) = \sum_{l=1}^{4} r_{i,j,l} e^{\rho_l u}, \quad i, j = 1, 2.$$

The coefficients, $r_{i,j,l}$, are given by

$$\begin{pmatrix} r_{1,1,l} & r_{1,2,l} \\ r_{2,1,l} & r_{2,2,l} \end{pmatrix} = \frac{(\rho_l + \mu_1)(\rho_l + \mu_2)}{\prod_{l=1, l \neq l}^{4}(\rho_l - \rho_l)} \boldsymbol{L}_c^*(\rho_l), \quad l = 1, 2, 3, 4,$$

where $\boldsymbol{L}_c^*(s)$ is the adjoint matrix of $\boldsymbol{L}_c(s)$.

The second approach to tackle $\mathbf{v}(u)$ is also applicable here. Note that there are two roots of equation (7.2) with positive real parts. For each of them, say, ρ_1 and ρ_2, we can compute the special case of the Gerber-Shiu function with penalty function $w_i(x,y) = e^{-\rho_i y}, i = 1,2$ in the Markov-modulated risk model without any dividend strategy. That is,

$$\theta_i(u) = \sum_{j_1=1}^{2} \alpha_{j_1} \sum_{j_2=1}^{2} \theta_{i,j_1,j_2}(u)k \quad u \geq 0, i \in E,$$

where the $\theta_{i,j_1,j_2}(u)$ is the (j_1,j_2)th element of the following matrix,

$$\mathbb{E}\left[e^{-\delta\tau-\rho_i|U(\tau)|} I(\tau < \infty)|U(0) = u\right], \quad u \geq 0, i \in E.$$

The initial value $\theta_i(0)$ can be solved from (3.36) and $\theta_i(u)$ can be obtained by following the decomposition steps of the expected discounted penalty function discussed in Li and Lu (2007).

The premium rates for different layers and the threshold levels are set to be the same in Example 1. We also set $\lambda_1 = 1$, $\lambda_2 = 0.4$, $\alpha_{12} = 1/4$, $\alpha_{21} = 3/4$, $\beta_1 = 1$ and $\beta_2 = 2$. Following the recursive algorithm in Chapter 4, we have the ruin probabilities under the two-threshold (three-layer) Markov-modulated risk model illustrated in Figure 7.3, and the numerical expressions for $\vec{\varphi}(u)$, $\vec{\varphi}_1(u;B)$, $\vec{\varphi}_2(u;B)$ and $\vec{\varphi}_3(u;B)$ in terms of linear combinations of exponential functions in the following forms:

- no dividend strategy,

$$\vec{\varphi}(u) = \begin{pmatrix} 0.6120 & 0.0029 \\ 0.3827 & 0.5083 \end{pmatrix} \begin{pmatrix} e^{-0.3903u} \\ e^{-1.7779u} \end{pmatrix}, \quad u \geq 0;$$

- one-threshold dividend strategy,

$$\vec{\varphi}_1(u;B_1) = \begin{pmatrix} 0.6025 & 0.0029 & 6.3775\times10^{-6} & 0.0156 \\ 0.3767 & 0.0574 & 1.3179\times10^{-5} & 0.0156 \end{pmatrix} \begin{pmatrix} e^{-0.3903u} \\ e^{-1.7779u} \\ e^{0.8824u} \\ 1 \end{pmatrix},$$

$$0 \leq u < 5,$$

$$\vec{\varphi}_2(u;B_1) = \begin{pmatrix} 0.5824 & -0.0482 \\ 0.3904 & -0.9360 \end{pmatrix} \begin{pmatrix} e^{-0.3491u} \\ e^{-1.7651u} \end{pmatrix}, \quad u \geq 5;$$

- two-threshold dividend strategy,

$$\vec{\varphi}_1(u;B_2) = \begin{pmatrix} 0.6000 & 0.0028 & 6.3615\times10^{-6} & 0.0196 \\ 0.3752 & 0.0572 & 1.3146\times10^{-5} & 0.0196 \end{pmatrix} \begin{pmatrix} e^{-0.3903u} \\ e^{-1.7779u} \\ e^{0.8824u} \\ 1 \end{pmatrix}, \quad 0 \le u < 5,$$

$$\vec{\varphi}_2(u;B_2) = \begin{pmatrix} 0.5800 & -0.0480 & 6.3615\times10^{-9} & 0.0041 \\ 0.3887 & -0.9321 & -1.3603\times10^{-8} & 0.0041 \end{pmatrix} \begin{pmatrix} e^{-0.3491u} \\ e^{-1.7651u} \\ e^{0.9603u} \\ 1 \end{pmatrix}, \quad 5 \le u < 10,$$

$$\vec{\varphi}_3(u;B_2) = \begin{pmatrix} 0.4453 & 0.1771 \\ 0.3202 & 0.1317 \end{pmatrix} \begin{pmatrix} e^{-0.3015u} \\ e^{-1.7509u} \end{pmatrix}, \quad u \ge 10.$$

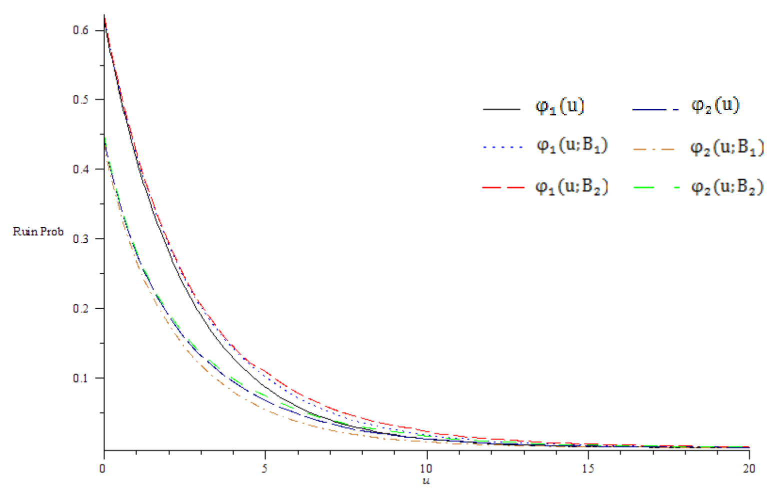


Figure 7.3: Ruin probabilities in the Markov-modulated risk model under different dividend strategies

As illustrated in Figure 7.3, it is not surprising that, if viewed separately, the ruin probabilities in state 2 are lower than those in state 1 under the same dividend strategies because of the fact that less claim arrivals and smaller claim amounts are expected in state 2.

Example 4 (*Expected Discounted Dividend Payments in the MAP Risk Model*)

In this example, we further look at the numerical results for the expected present value of the total dividend payments prior to ruin. Similar to Example 3, claims occur under two conditions. While under the abnormal condition, there is a probability p for the claim arrival process to return back to the normal condition. The claim amounts are related to the phase transition. For example, the distribution of the claim amounts from phase 1 to phase 2 is exponentially distributed with mean $1/\beta_{12}$. With $m = 2$, we have

$$\boldsymbol{D_1} = \begin{pmatrix} \lambda_1 & 0 \\ p\lambda_2 & (1-p)\lambda_2 \end{pmatrix}, \quad \boldsymbol{D_0} = \begin{pmatrix} -\alpha_{12} - \lambda_1 & \alpha_{12} \\ \alpha_{21} - p\lambda_2 & -\alpha_{21} - (1-p)\lambda_2 \end{pmatrix}.$$

For the claim amounts matrix, we have

$$\boldsymbol{f}(x) = \begin{pmatrix} \beta_{11} e^{-\beta_{11} x} & \beta_{12} e^{-\beta_{12} x} \\ \beta_{21} e^{-\beta_{21} x} & \beta_{22} e^{-\beta_{22} x} \end{pmatrix}.$$

In this case, matrix $\boldsymbol{L}_c(s)$ is given by

$$\boldsymbol{L}_c(s) = \frac{1}{c} \begin{pmatrix} cs - \delta - (\alpha_{12} + \lambda_1) + \frac{\lambda_1 \beta_{11}}{s + \beta_{11}} & \frac{\alpha_{12}\beta_{12}}{s+\beta_{12}} \\ \alpha_{21} - p\lambda_2 + \frac{p\lambda_2\beta_{21}}{s+\beta_{21}} & cs - \delta - (\alpha_{21} + (1-p)\lambda_2) + \frac{\lambda_2\beta_{22}}{s+\beta_{22}} \end{pmatrix}.$$

For $\delta > 0$, equation $\det[\boldsymbol{L}_c(s)] = 0$ has two positive real roots which we denote by ρ_1 and ρ_2. The approach to evaluate $\mathbf{v}(u)$ is similar to the one in Example 3.

To illustrate it numerically, we set $p = 0.5$, $\beta_{11} = 1$, $\beta_{12} = 1$, $\beta_{21} = 0.5$ and $\beta_{22} = 0.5$. In the case, the claims occurring in the same phase follow the same claim amount distribution, regardless the fact that they will transit to other states or not. Also we set the threshold level to $B = \{0, 20, 40, \infty\}$, the premium rates for three layers to $c_1 = 163.5$, $c_2 = 133.5$ and $c_3 = 103.5$, and the discounted valuation factor δ to 0.1. Table 7.1 displays the expected discounted dividend payments prior to ruin in the MAP risk model with a one-threshold (two-layer) strategy and with a two-threshold (three-layer) strategy for the same initial surplus values. It is interesting to see that, when $u \le 40$, the discounted dividend payments

Table 7.1: Expected discounted dividend payments in the multi-threshold MAP risk model

u	Three layers	Two layers
0	152.71	311.33
10	321.06	671.02
20	347.29	730.80
30	355.21	792.82
40	423.23	821.51
50	871.54	846.15
60	937.18	855.69

in the three-layer model are smaller than the one in the two-layer model with the same initial surplus. When the initial surplus u is higher than the second (last) threshold, the expected discounted dividend payments in the three-layer model are higher. It is the same case as the Strategy 3 and Strategy 4 in Table 3 in Badescu et al. (2007), though different parameters are used. We explain it as the situation that when the multi-threshold strategy is involved, the ruin probability in this model would increase to some extend such that ruin occurs earlier than the one in the model with a one-threshold strategy only. If it is the case, the expected discounted dividend payments are smaller. When the surplus process starts at a high initial level, for example, above b_2 in the example, larger dividend payments in the multi-threshold risk model are paid out immediately, leading to a larger expected discounted dividend payments in total.

Example 5 (*"Contagion" Example*)

In this example, we illustrate the applicability of our results from the differential approach by examining the "contagion" risk model introduced in Badescu et al. (2005). This contagion model assumes that there are two types of claims here. One is the standard claims which occur according to a Poisson process at rate $\lambda_1 = 1$ all the time. The other one is the additional infectious claims which occur at rate $\lambda_2 = 10$ during the contagion periods, say the abnormal state B. Standard claim amounts have mean $1/\beta_1 = 1/5$, while the infectious claim amounts have mean $1/\beta_2 = 3$. It is also assumed that the rate at which the system switches from A to B is $\alpha_A = 0.02$ and the rate at which the system switches from B to A is $\alpha_B = 1$. So the stationary distribution for state A is $\alpha_B/(\alpha_A + \alpha_B)$ and for state B is $\alpha_A/(\alpha_A + \alpha_B)$. Following the parameters from Badescu et al. (2007), we suppose that premiums are collected at rates 2, 1.5 and 1 for the first, second and third layers, respectively, and the corresponding threshold levels are located at $B = (0, 20, 40, \infty)$.

Table 7.2: Expected discounted dividend payments in "contagion" example

u	$\delta = 0.1$	$\delta = 0.01$	$\delta = 0.001$	Badescu et al. (2007), $\delta = 0.001$
0	158.99	323.23	356.68	N/A
10	350.55	457.58	500.95	503.00
30	417.19	671.02	692.82	692.60
50	688.25	802.29	821.50	842.07
70	814.98	926.93	942.78	968.82

Table 7.2 displays the expected discounted dividend payments prior to ruin under different settings for δ, including the results from Badescu et al. (2007) with $\delta = 0.001$ for the comparative purpose. Recall that the MAP risk model in Badescu et al. (2005) is studied through fluid queues, while our results come from the differential recursive approach. As we can see from Table 7, with the same parameters, the expected discounted dividend

payments by using two different methods are relatively close to each other. When δ is larger, the expected discounted dividend payments are smaller for the same initial surplus as expected.

Chapter 8

Conclusion

In this thesis, we have examined the expected discounted penalty function and the distribution of the total dividend payments prior to ruin in the multi-threshold MAP risk model. It is seen that the differential approach is applicable to derive systems of integro-differential equations for the discounted penalty function and the moments of the dividend payments prior to ruin. Analytical solutions are provided by Theorem 1 and computations can follow the corresponding algorithms. Also it is illustrated in numerical examples that with different settings on transition matrices, the multi-threshold MAP risk model has an extensive flexibility in modeling the claim arrivals and the claim amounts, including the compound Poisson risk model, the Sparre Andersen risk model and the Markov-modulated risk model as special cases.

In addition to the differential approach, we have considered an alternative recursive method which is called the layer-based approach for the expected discounted dividend payments prior to ruin. It is a generalization from the one used in the classical compound Poisson risk model in Albrecher and Hartinger (2007). The complete solution of the expected discounted dividend payments for the k-layer MAP risk model is shown to be in the form of the solution for the classical one-layer model and the solution for the $(k-1)$-layer model with the lower layers coincidence.

In reality, fitting data into the MAP risk model can be a big challenge. The moment matching technique is computationally more efficient but somewhat restrictive because it mainly deals with two-state Markov-modulated Poisson process. See Guselia (1991). The MLE-based technique is applicable and has been developed by Horvath et al. (2000). Also the so-called EM algorithm has proved to be a good means of approximating the maximum likelihood estimator and been used as an estimation procedure for the MAP model. See, for instance, Breuer (2002). Another challenge comes from a possible hidden Markov model. Indeed, the states described as high risks and low risks, for instance, are not directly visible and we need to resort to the Baum-Welch algorithm to find the unknown parameters of a hidden Markov model. Detailed discussion can be referred to Welch (2003).

Even though the method presented in this thesis can be applied to the MAP risk model with any arbitrary threshold setting, we do not claim to answer the question of optimal dividend strategy, in which the ultimate goal is to maximize the expectation of the discounted dividends. For the classical compound Poisson risk model, such a strategy is discussed extensively in Gerber and Shiu (2006) and is implemented by later literature. In certain cases, the optimal strategy for the classical compound Poisson risk model is obtained in the constant barrier strategy. In Albrecher and Hartinger (2006), it is shown that the optimality of horizontal barrier strategies does not carry over to the Sparre Andersen models in general and a counter-example is provided.

Another limitation of our model is that the multi-threshold dividend strategy in terms of the threshold levels does not depend on the external environmental process $\{J(t); t \geq 0\}$. There is a practical motivation to adopt the dividend payment strategy that depends on the threshold levels. As remarked in Cheung and Landriault (2009), it is reasonable for the insurer to set a higher threshold level in the periods of the so-called dangerous state, such that dividends are paid only if the surplus reaches a more secure level and more capital is available to face the adverse claims experience.

Appendix A

Some Sample Paths

Figure A.1: Sample path of the surplus process under the constant barrier dividend strategy

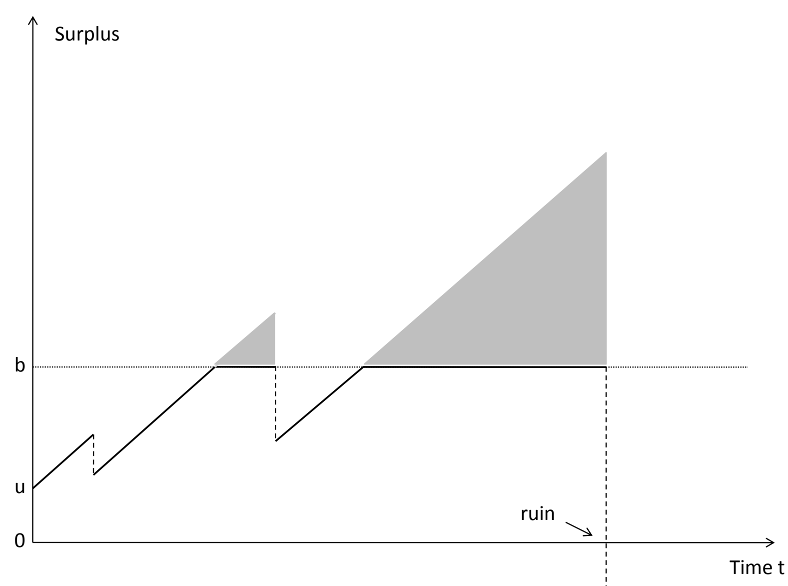


Figure A.2: Sample path of the surplus process under the threshold dividend strategy

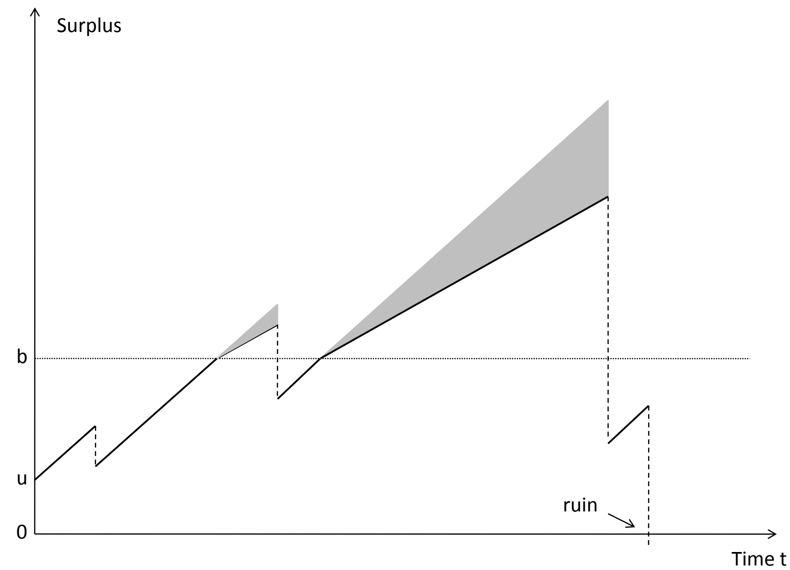


Figure A.3: Sample path of the surplus process under the multi-threshold dividend strategy

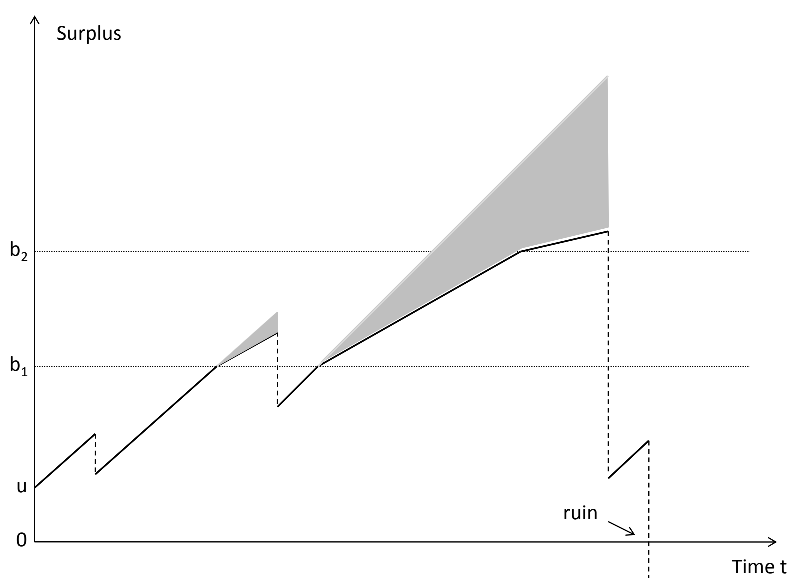


Figure A.4: Sample path 1 of the surplus process $U_B(t)$ to reach b

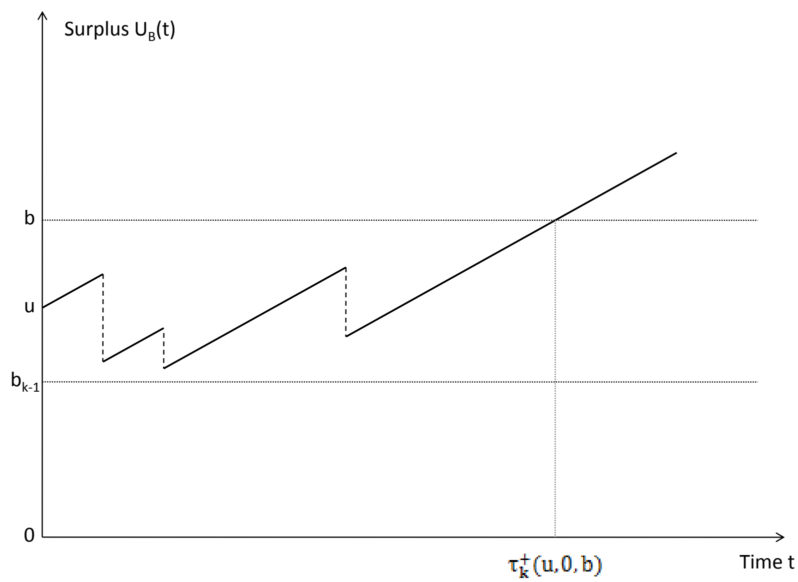


Figure A.5: Sample path 2 of the surplus process $U_B(t)$ to reach b

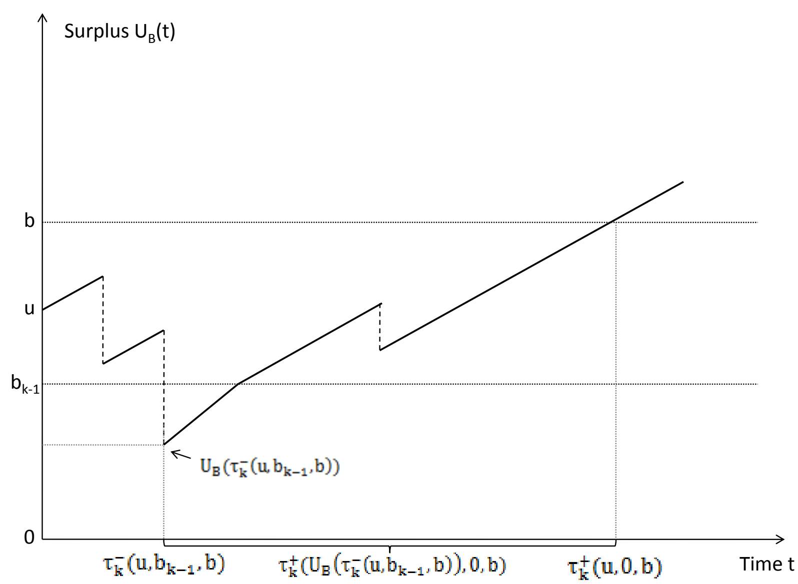


Figure A.6: Sample path 1 of the surplus process $U_{1,k}(t)$ to reach $b - b_{k-1}$

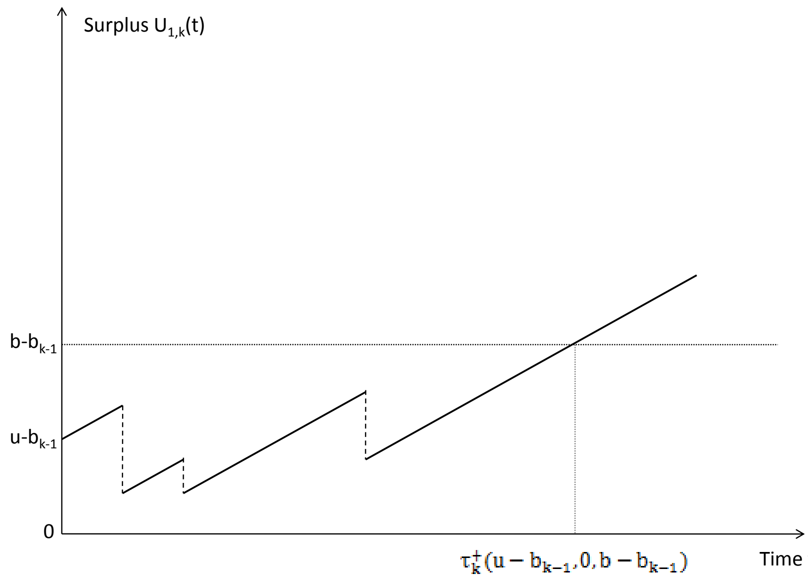


Figure A.7: Sample path 2 of the surplus process $U_{1,k}(t)$ to reach $b - b_{k-1}$

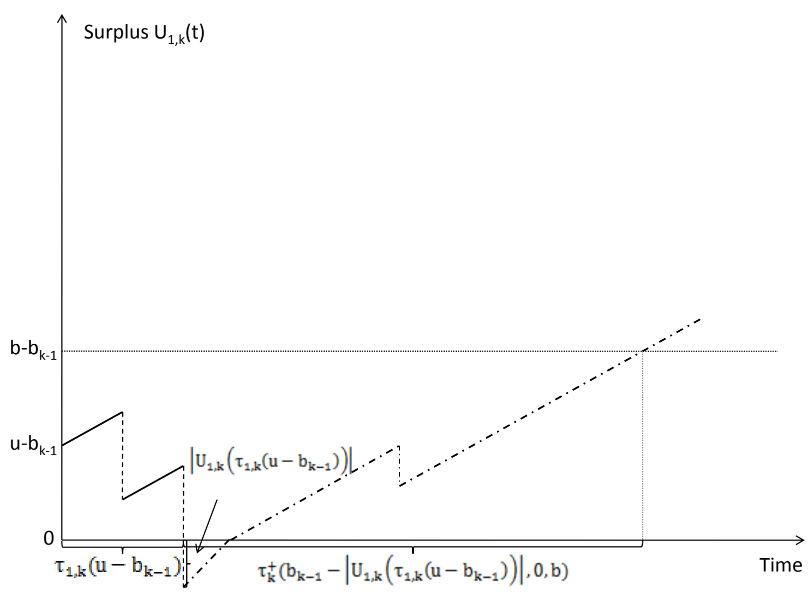
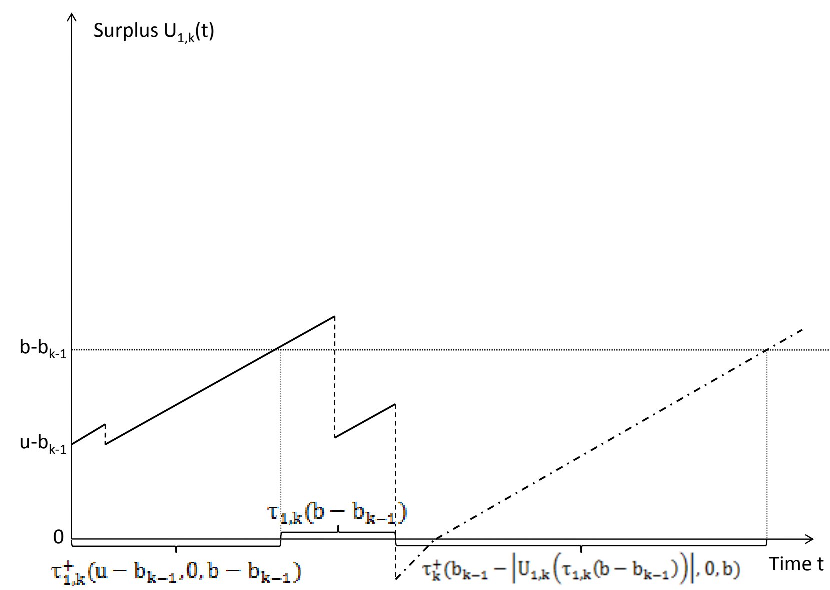


Figure A.8: Sample path 3 of the surplus process $U_{1,k}(t)$ to reach $b - b_{k-1}$



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