Predictive Estimation in Canadian Federal Elections

by

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Abstract

Various estimation methods are employed to provide seat projections during Canadian federal elections. This project explores discrepancies between the real outcomes of recent Canadian federal elections and the predictions by the existing approaches such as the ones proposed by Grenier (www.threehundredeight.com) and Rosenthal (2011). It appears that each seat projection procedure requires a set of assumptions, but the assumptions are not explicitly listed in the accessible references. We formulate the required assumptions used in the two prediction procedures proposed by Rosenthal (2011), and present variance estimation procedures. Departures from the assumptions are explored with real data from the 2006, 2008, 2011, and 2015 federal election. An extensive simulation study is conducted to examine potential impacts of various deviations from the assumptions. The simulation indicates that, compared to other assumption violations, misleading polling results may cause the most damage to the prediction. In addition, we find by the simulation that the prediction is least affected by a change in number of voters and the heterogeneity of riding patterns within a region may not affect the the prediction at the national level.

Keywords: assumption violation; descriptive analysis; seat prediction; simulation study; variance estimation

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Chapter 1

Introduction

1.1 Background

Canada is a democracy which chooses governments by electing representatives to Parliament. Under certain restrictions, anyone is allowed to run for seats in the House of Commons. The only requirements of becoming a candidate in the Canadian federal elections are to be a Canadian citizen, to be at least 18 years old on the election day, and to file official documents[1]. Under these restrictions, almost anyone could become a candidate.

The House of Commons currently consists of 338 seats which correspond to 338 electoral districts (also known as “ridings”) in Canada. There are 335 ridings spread across the 10 provinces and 1 riding in each of the 3 territories in Canada. The candidate who wins in a riding will be awarded with the corresponding seat in the House of Commons. The winner in a riding is also known as the “Member of Parliament” for that specific riding.

Canada uses an electoral system called “single member plurality”[2]. To put it simply, under this system, the candidate who receives the most votes in a riding wins the corresponding seat in the House of Commons. Most candidates are affiliated with one of the five major political parties in Canada (Conservative Party of Canada, Liberal Party of Canada, New Democratic Party, Green Party of Canada, and Bloc Québécois). If the members of a party win more than half of the total number of seats, that party will form what is known as a “majority government”. The advantage of a majority government is that the party can pass legislation much more easily by maintaining the confidence of the House of Commons[3].

[1] http://www.elections.ca/content.aspx?section=pol&dir=can/bck&document=index&lang=e
[2] https://en.wikipedia.org/wiki/First-past-the-post_voting
[3] canadaonline.about.com/od/elections/g/majority.htm

With this in mind, accurately predicting election outcomes in the lead up to election day is useful to many people. For voters, they may change their voting behaviour based on the likelihood of each candidate of winning the election. Some voters may choose to vote for other candidates to prevent a certain party from forming a majority government. For candidates, it is important for them to know how likely it is for them to win in the election. Based on the prediction, they may choose to change their strategy of obtaining votes from the voters. For political parties, it may be important for them to know how many seats they can expect in the election. Based on the prediction, they would be able to determine whether they should continue with the same strategy or change to a new strategy.

1.2 Motivation

Many methods have been used to predict the outcome of an election. The most naive way of obtaining a prediction is simply to assume that the proportion of seats obtained in the election is going to be equal to the estimated proportion of votes. However, Rosenthal points out that “a party’s percentage of seats of often very different from its overall percentage of votes” (Rosenthal, 2011). In the same paper, Rosenthal proposes two similar, but different, estimators to improve the accuracy of the prediction of an election outcome. What Rosenthal proposes are what he calls a “uniform voter-shift adjustment” and a “ratio voter adjustment”. Éric Grenier, the owner of the seat prediction website, threehundredeight.com, also proposes an estimator for predicting the riding and election outcomes[5]. Both estimators require several assumptions, which may or may not be true in a real election. Due to a lack of information on Grenier’s estimator and inaccessibility of the related literature, this project focuses on examining the performance of Rosenthal’s estimators [6] if the assumptions are violated.

Comparisons between the actual election outcome and the estimates obtained based on the estimators proposed by Rosenthal and Grenier for the 2011 and 2015 Canadian federal election are shown below.

Table 1.1: The actual outcome of the 2011 Canadian federal election against predicted ones. Absolute difference between actual and predicted outcomes shown in parentheses.

	Election Outcome	Grenier	Rosenthal (Uniform)	Rosenthal (Ratio)
Conservative	166	143 (23)	157 (9)	160 (6)
Liberals	34	60 (26)	40 (6)	33 (1)
NDP	103	78 (25)	104 (1)	103 (0)
Green	1	0 (1)	0 (1)	0 (1)
Bloc	4	27 (23)	7 (3)	12 (8)
Others	0	0 (0)	0 (0)	0 (0)

Table 1.2: The actual outcome of the 2015 Canadian federal election against predicted ones. Absolute difference between actual and predicted outcomes shown in parentheses.

	Election Outcome	Grenier	Rosenthal (Uniform)	Rosenthal (Ratio)
Conservative	99	118 (19)	140 (41)	144 (45)
Liberals	184	146 (38)	121 (63)	131 (53)
NDP	44	66 (22)	74 (30)	56 (12)
Green	1	1 (0)	1 (0)	1 (0)
Bloc	10	7 (3)	1 (9)	1 (9)
Others	0	0 (0)	1 (1)	5 (5)

As seen in table 1.1, Rosenthal provides a very good prediction for the 2011 election. Both estimators proposed by Rosenthal were fairly close to the real outcome of the 2011 election. However, the two Rosenthal's estimates were very different than the real outcome for the 2015 election (see table 1.2). Grenier's estimator provided an estimate better than Rosenthal's estimators, but it is still very different than the real election outcome. This has motivated us to explore how to improve the existing predictive estimation procedures in the federal election.

We aim to understand what can lead to poor performance of the existing approaches in the prediction of Canadian federal elections. Why does an estimator that worked so well in one election fail to provide a good estimator in the following election?

1.3 Organization of Project

The project is organized as follows. Chapter 1 provides background information. Chapter 2 introduces notations and introduces Grenier's and Rosenthal's estimators, and an approximate variance of each of Rosenthal's estimator is presented. Chapter 3 includes an exploratory data analysis, design, and results of the simulation study. Chapter 4 includes the conclusion and thoughts about future research.

Chapter 2

Existing Approaches

2.1 Notation

This section introduces the notation used throughout the rest of this project.

Indices

t Index for the t^{th} Canadian federal election; $t = 39,\ldots,42$

$$t = \begin{cases} 39 & \text{for the 2006 Canadian federal election} \\ 40 & \text{for the 2008 Canadian federal election} \\ 41 & \text{for the 2011 Canadian federal election} \\ 42 & \text{for the 2015 Canadian federal election} \end{cases}$$

i Index for ridings; i=1,...,$I(t)$, where

$$I(t) = \begin{cases} 308 & \text{if } t = 39,40,41 \\ 338 & \text{if } t = 42 \end{cases}$$

j Index for party; j=1,...,6, where

$$j = \begin{cases} 1 & \text{if Conservative Party of Canada} \\ 2 & \text{if Liberal Party of Canada} \\ 3 & \text{if New Democratic Party} \\ 4 & \text{if Green Party of Canada} \\ 5 & \text{if Bloc Québécois} \\ 6 & \text{if Others} \end{cases}$$

l Index for region: l = 1,...,5, where

$$l = \begin{cases} 1 & \text{if riding } i \text{ is in the Atlantic Provinces} \\ 2 & \text{if riding } i \text{ is in Quebec} \\ 3 & \text{if riding } i \text{ is in Ontario} \\ 4 & \text{if riding } i \text{ is in the Prairies or the Territories} \\ 5 & \text{if riding } i \text{ is in British Columbia} \end{cases}$$

k Index for poll in reverse chronological order; k = 1,...,K(t), where K(t) is the total number of polls conducted for election t

Election Results

Y_{jt} The number of seats obtained by party j in the t^{th} election

v_{ijt} The number of votes obtained by party j in riding i in election t

v^*_{jlt} The number of votes obtained by party j in region l in election t

v_{jt} The number of votes obtained by party j in election t

n_{it} The number of voters who voted in riding i in election t

n^*_{lt} The number of voters who voted in region l in election t

p_{ijt} The probability of an individual voting for party j in riding i in election t

p^*_{jlt} The probability of an individual voting for party j in region l in election t

p_{jt} The probability of an individual voting for party j in election t

r_{ijt} The proportion of votes obtained by party j in riding i in election t

r^*_{jlt} The proportion of votes obtained by party j in region l in election t

r_{jt} The proportion of votes obtained by party j in election t

Poll Information

q^*_{jklt} The proportion of people in region l who indicated they will vote for party j in poll k for election t

q_{jkt} The proportion of people who indicated they will vote for party j in poll k for election t

s_{kt} The sample size of poll k for election t

s^*_{klt} The sample size of poll k from region l for election t

M^*_{jklt} The number of people from region l who indicate that they will vote for party j in election t in poll k

M_{jkt} The number of people who indicate that they will vote for party j in election t in poll k

Others

τ_t The total number of seats in the House of Commons in the t^{th} Canadian federal election

c_{ijt} Indicator for whether party j won in riding i in the t^{th} election or not

$$c_{ijt} = \begin{cases} 1 & \text{if } c_{ijt} = \max_j\{c_{ijt}\} \\ 0 & \text{Otherwise} \end{cases}$$

ν_{it} The number of eligible voters in riding i in election t

As with conventional statistics practice, the "hat" symbol $(\hat{\ })$ is used to denote an estimated quantity. For example, $\hat{p}_{jt}$ represents the estimated probability of an individual voting for party j in election t. The dot symbol $(\cdot)$ is used to denote a sum over the specific index. For example, $n_{\cdot t}$ represents the total number of voters across all ridings.

2.1.1 Relationships Between Variables

There are a few relationships between the variables explained above that are crucial to keep in mind.

$$n_{it} = \sum_{j=1}^{6} v_{ijt} = v_{i\cdot t} \tag{2.1}$$

Since v_{ijt} denotes the number of votes obtained by party j in riding i in election t, the total number of voters in riding i who voted in election t can be represented as the sum of the number of votes obtained by each party in riding i

$$n_{it} \leq \nu_{it} \tag{2.2}$$

To practice the right to vote of voluntary rather than mandatory. Some voters may choose to not vote for one reason or another. Therefore, the total number of voters who voted in riding i in election t can only be less than or equal to the number of eligible voters in riding i in election t.

$$Y_{jt} = c_{\cdot jt} = \sum_{i=1}^{I(t)} c_{ijt} \tag{2.3}$$

If the winning party is known for each riding, then the total number of seats for each party can be obtained by finding the total number of ridings in which the respective parties won.

$$\tau_t = Y_{\cdot t} = \sum_{j=1}^{6} Y_{jt} = c_{\cdot\cdot t} \tag{2.4}$$

The total number of seats in the house of commons can be obtained by finding the sum of the number of seats that each party received.

$$r_{ijt} = \frac{v_{ijt}}{n_{it}} \tag{2.5}$$

The proportion of votes that each party obtained in riding i in election t can be obtained by finding the ratio between the number of votes that party received in riding i and the number of people who voted in riding i.

2.2 Grenier's Approach

Éric Grenier is the owner of the seat projection website www.threehundredeight.com. He uses his estimator to predict the outcome of an election. Not every detail of the estimator used by Grenier is explicitly stated, but there are several things to note about the estimator[1]. First, it estimates the probability of a voter voting for party j in region l, p^*_{jlt}, by using all available polls and weighs the polls by their age, margin of error, and the track record of the polling firm. The weight of the poll is decreased by 35% for each day since the last day it has been in the field. The polls are also weighed based on the margin of error. If the lowest margin of error in the polls is MOE_L, and MOE_k denotes the margin of error for poll k, then the weight for poll k is MOE_L/MOE_k. The polls are also weighed based on the track record of the polling firm. The calculation of the track record of the polling firm is not explicitly stated, but it measures how accurate the polling firm was in the past 10 years.

After obtaining an estimate for the probability that an individual would vote for party j in region l, Grenier combines the estimated probability with several other factors such as incumbency, leaders, star candidates, and the presence of independents and estimates the number of votes each party would obtain based on the estimated regional probability of voting for party k and the results of the three previous elections.

$$\hat{p}^*_{jlt} = \sum_{k=1}^{K(t)} w_{kt} q^*_{jklt} \tag{2.6}$$

where K(t) represents the number of available polls at time t.

$$\hat{p}_{ijt} = \left[\frac{1}{2} \left(\frac{\hat{p}^*_{jlt}}{r^*_{jl,t-1}} r_{ij,t-1} \right) + \frac{1}{3} \left(\frac{\hat{p}^*_{jlt}}{r^*_{jl,t-2}} r_{ij,t-2} \right) + \frac{1}{6} \left(\frac{\hat{p}^*_{jlt}}{r^*_{jl,t-3}} r_{ij,t-3} \right) \right] \cdot f_{ijt} \tag{2.7}$$

w_{kt} in equation (2.6) depends on three different factors. It is calculated based on the age, margin of error, and track record of the polling firm for the k^{th} poll in election t. q^*_{jklt} is the sample proportion of voters in region l in the k^{th} poll for election t who indicated they will vote for party j. f_{ijt} in equation (2.7) calculated based on the incumbency, leaders, star candidates, presence of independent candidates, as well as other variables for party j in riding i in election t. Only the calculation of the incumbent effect is explicitly stated. If the candidate is not an incumbent for election t, then $\hat{p}_{ijt}$ is penalized by 10% of what it should have been if the candidate is an incumbent.

Since $\hat{p}_{ijt}$ is an estimate of the probability of an individual voting for party j in riding i, it must add up to 1 in each riding. In other words, $\sum_j \hat{p}_{ijt} = 1$ for all i=1,…,$I(t)$. If

[1] http://www.threehundredeight.com/2015/01/introducing-2015-federal-election.html

$\sum_j \hat{p}_{ijt}$ is not 1, then $\hat{p}_{ijt}$ will increase or decrease proportionally until $\sum_j \hat{p}_{ijt}$ is equal to 1. The estimator proposed by Grenier includes many variables and would require extensive research to be of use. The track record of different polling firms would require a lot of poll results from past elections and the polls may not be readily available anymore. Even if the polls are still available, it would be very difficult for anyone other than Grenier to calculate.

Also, Grenier's estimator can be expressed in the following way.

$$\hat{p}_{ijt} = \left[\frac{1}{2}\left(\frac{\hat{p}^*_{jlt}}{r^*_{jl,t-1}} r_{ij,t-1}\right) + \frac{1}{3}\left(\frac{\hat{p}^*_{jlt}}{r^*_{jl,t-2}} r_{ij,t-2}\right) + \frac{1}{6}\left(\frac{\hat{p}^*_{jlt}}{r^*_{jl,t-3}} r_{ij,t-3}\right)\right] \cdot f_{ijt} \tag{2.8}$$

$$= \hat{p}^*_{jlt}\left(\frac{r_{ij,t-1}}{2r^*_{jl,t-1}} + \frac{r_{ij,t-2}}{3r^*_{jl,t-2}} + \frac{r_{ij,t-3}}{6r^*_{jl,t-3}}\right) \cdot f_{ijt} \tag{2.9}$$

The estimated probability of a voter voting for party j in the current election in region l is multiplied by a ratio that is based on the previous three election riding and regional results as well as a function which includes the incumbency status of the individual candidates for party j.

It is worth noting that the estimates based on Grenier's estimator cannot be replicated due to a lack of details on his website, www.threehundredeight.com, and references and supports is inaccessible in the literature.

2.3 Rosenthal's Approaches (Rosenthal, 2011)

Shortly after the Canadian federal election in 2011, Rosenthal proposed the following two estimators to show that the "surprising" result of the election should have, in fact, been predictable. In the 2011 federal election, the Conservatives won a majority government. The Conservatives majority was not considered possible by the media (Rosenthal, 2011).

$$w_{kt} = \frac{s_{kt}}{\sum_{k=1}^{9} s_{kt}} \tag{2.10}$$

Equation (2.10) shows that Rosenthal's weighting is based only on the sample size of the individual polls. Rosenthal did not use the age of the poll and the track record of the polling firm when calculating the weights.

$$\hat{p}^*_{jl(i)t} = \sum_{k=1}^{9} w_{kt} q^*_{jkl(i)t} \tag{2.11}$$

The biggest difference between the estimators used by Rosenthal and Grenier is that Rosenthal used the 9 most recent polls when he is estimating, $p^*_{jl(i)t}$, the probability of a voter voting for party j in each region l. This is different than Grenier, who uses every poll available. Also, Rosenthal weighs the polls based on the sample size and disregards the age of the poll and the track record of the polling firm.

$$\hat{v}_{ijt,U} = v_{ij,t-1} + \left(\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})\right) \cdot n_{i,t-1} \tag{2.12}$$

Equation (2.12) is what Rosenthal calls a "uniform voter-shift adjustment". The term $(r_{j,t-1} - \hat{p}_{j,t-1})$ is what Rosenthal calls the "overperforming effect", which measures the difference between the proportion of votes that a party received and the estimated proportion of votes that a party was expected to obtain in election $t-1$. Equation (2.12) may look complicated, but it is very simple intuitively. This estimator begins with the number of votes party j received in the last election and add or subtracts from that number based on the current estimated proportion of votes party j would receive. If party j is estimated to receive a higher proportion of votes in the upcoming election, intuitively, they should be expected to receive more votes than the last election. Conversely, if party j is estimated to receive a lower proportion of votes in the upcoming election, they should be expected to receive less votes than they did in the last election.

By including the "overperforming effect", Rosenthal is implying that even the best polling firm will capture a systematically different population in their polls compared to the population who actually votes on election day. Rosenthal includes the "overperforming effect" and adds it to the estimated probability, $\hat{p}^*_{jlt}$, to obtain an estimate for the proportion of votes that each party will receive in the election.

For example, assume that in the last election, the proportion of votes received by party j is exactly the same as the proportion of votes it is estimated to receive, such that the "overperforming effect" is 0. Further suppose 10,000 people voted in that riding in the last election, of which 2500 voted for party j, and party j received 30% of the votes in region $l(i)$. Lastly, suppose that party j is estimated to receive 40% of the votes in region $l(i)$ in the upcoming election. Based on this estimator, party j would be expected to receive

$$\hat{v}_{ijt,U} = 2500 + (0.40 - 0.30 + 0) \cdot 10000 = 3500$$

Therefore, based on the uniform estimator, party j would be estimated to obtain 3500 votes in riding i in the upcoming election.

Rosenthal also proposed what he calls a "ratio voter adjustment", which is shown in equation (2.13).

$$\hat{v}_{ijt,R} = v_{ij,t-1} \frac{\hat{p}^{*}_{jl(i)t} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r^{*}_{jl(i),t-1}} \tag{2.13}$$

Similar to the "uniform voter-shift adjustment", the "ratio voter adjustment" contains "overperforming effect". Equation (2.13) provides very intuitive estimates. It starts with the votes obtained by party j in riding i in the last election and then finds the ratio between the proportion of votes party j is estimated to obtain in region $l(i)$ and the proportion of votes party j actually obtained in region $l(i)$ in the last election. If party j is estimated to obtain a higher proportion of votes than the last election, then this ratio will be greater than 1, which leads to a higher estimated number of votes compared to last election. If the opposite is true, that party j is estimated to obtain a lower proportion of votes than the last election, then this ratio will be less than 1, which will lead to a lower estimated number of votes compared to last election. Using the same numbers as the previous example, we obtain the following estimate:

$$\hat{v}_{ijt,R} = 2500 \frac{(0.40 + 0)}{0.30} = 3333.33$$

Based on the ratio estimator, party j would be estimated to obtain 3333 votes in riding i in the upcoming election.

The biggest difference between the estimators proposed by Rosenthal compared to Grenier's estimator is that Rosenthal's estimators include the "overperforming effect" term. To the best of the author's knowledge, Rosenthal's estimators are the first estimators to try to adjust for the systematic difference between the population captured by the polls and the population who actually votes. The "overperforming effect" Rosenthal uses implies that the voting pattern of the non-voters is different than the voting pattern of the voters.

After obtaining an estimate of the number of votes for all ridings and for all parties, the winner of each riding is predicted to be the candidate with the highest estimated number of votes in his or her respective riding. After the list of winners is estimated, the number of seats that each party can expect to obtain can be estimated using the list of predicted winners. The method used by Rosenthal is able to provide very good prediction for the 2011 Canadian federal election; however, it is not able to provide a good prediction for the 2015 Canadian federal election.

2.4 Variance of Rosenthal's Estimators

The variance of Rosenthal's estimators are derived using multiparameter delta method[7] and are shown below. In this section, X_{ijta} denotes the a^{th} component of $\mathbf{X_{ijt}}$, and A represents the total number of components in $\mathbf{X_{ijt}}$. Also, the number of voters, n_{it}, and the number of individuals sampled in region l, s_l, are assumed to be constants. The derivations can be found in Appendix A.

Rosenthal's Uniform Estimator

$$Var(\hat{v}_{ijt,U}) \approx \sum_{a=1}^{A} \left\{ \left(\frac{\partial \hat{v}_{ijt,U}}{\partial X_{ijta}} \right)^2 Var(X_{ijta}) + \sum_{m<a} 2Cov(X_{ijtm}, X_{ijta}) \right\} \tag{2.14}$$

where,

$$\mathbf{X}_{ijt} = \begin{bmatrix} M^*_{j\cdot 1,t-1} \\ M^*_{j\cdot 2,t-1} \\ M^*_{j\cdot 3,t-1} \\ M^*_{j\cdot l(i)t} \\ v_{1j,t-1} \\ \vdots \\ v_{Ij,t-1} \end{bmatrix} \quad \frac{\partial \hat{v}_{ijt,U}}{\partial X_{ijta}} = \begin{bmatrix} -\frac{n_{i,t-1}}{\sum_l \sum_k s^*_{kl,t-1}} \\ -\frac{n_{i,t-1}}{\sum_l \sum_k s^*_{kl,t-1}} \\ -\frac{n_{i,t-1}}{\sum_l \sum_k s^*_{kl,t-1}} \\ \frac{n_{i,t-1}}{\sum_k s^*_{kl(i)t}} \\ \frac{\partial \hat{v}_{ijt,U}}{\partial \hat{v}_{1j,t-1}} \\ \vdots \\ \frac{\partial \hat{v}_{ijt,U}}{\partial \hat{v}_{Ij,t-1}} \end{bmatrix}$$

$$\frac{\partial \hat{v}_{ijt,U}}{\partial \hat{v}_{xj,t-1}} = \begin{cases} 1 - \frac{n_{i,t-1}}{\sum_{l(i)=l} n_{i,t-1}} + \frac{n_{i,t-1}}{\sum_i n_{i,t-1}} & \text{if } x = i \\ -\frac{n_{i,t-1}}{\sum_{l(i)=l} n_{i,t-1}} + \frac{n_{i,t-1}}{\sum_i n_{i,t-1}} & \text{if } l(x) = l(i) \text{ and } x \neq i \\ \frac{n_{i,t-1}}{\sum_i n_{i,t-1}} & \text{if } l(x) \neq l(i) \end{cases}$$

$$Var(X_{ijta}) = \begin{cases} \sum_{k=1}^{9} s^*_{k1,t-1} q^*_{jk1,t-1}(1 - q^*_{jk1,t-1}) & \text{if } a = 1 \\ \sum_{k=1}^{9} s^*_{k2,t-1} q^*_{jk2,t-1}(1 - q^*_{jk2,t-1}) & \text{if } a = 2 \\ \sum_{k=1}^{9} s^*_{k3,t-1} q^*_{jk3,t-1}(1 - q^*_{jk3,t-1}) & \text{if } a = 3 \\ \sum_{k=1}^{9} s^*_{kl(i)t} q^*_{jkl(i)t}(1 - q^*_{jkl(i)t}) & \text{if } a = 4 \\ n_{a-4,t-1} p_{a-4,j,t-1}(1 - p_{a-4,j,t-1}) & \text{if } a \geq 5 \end{cases}$$

$Cov(X_{ijtm}, X_{ijta})$ is assumed to be 0 when m $\neq$ a because each riding is considered to be independent of all other ridings.

Rosenthal's Ratio Estimator

$$Var(\hat{v}_{ijt,R}) \approx \sum_{a=1}^{A} \left\{ \left(\frac{\partial \hat{v}_{ijt,R}}{\partial X_{ijta}} \right)^2 Var(X_{ijta}) + \sum_{m<a} 2Cov(X_{ijtm}, X_{ijta}) \right\} \tag{2.15}$$

where,

$$\mathbf{X}_{ijt} = \begin{bmatrix} M^*_{j\cdot 1,t-1} \\ M^*_{j\cdot 2,t-1} \\ M^*_{j\cdot 3,t-1} \\ M^*_{j\cdot l(i)t} \\ v_{1j,t-1} \\ \vdots \\ v_{Ij,t-1} \end{bmatrix} \quad \frac{\partial \hat{v}_{ijt,R}}{\partial X_{ijta}} = \begin{bmatrix} -\frac{v_{ij,t-1} \sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1} \sum_l s^*_{\cdot l,t-1}} \\ -\frac{v_{ij,t-1} \sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1} \sum_l s^*_{\cdot l,t-1}} \\ -\frac{v_{ij,t-1} \sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1} \sum_l s^*_{\cdot l,t-1}} \\ \frac{v_{ij,t-1} \sum_{l(x)=l(i)} n_{x,t-1}}{s^*_{\cdot l(i)t} \sum_{l(x)=l(i)} v_{xj,t-1}} \\ \frac{\partial \hat{v}_{ijt,R}}{\partial \hat{v}_{1j,t-1}} \\ \vdots \\ \frac{\partial \hat{v}_{ijt,R}}{\partial \hat{v}_{Ij,t-1}} \end{bmatrix}$$

$$G = \frac{g_1}{g_2} = \frac{v_{ij,t-1} \sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1}}$$
$$H = \frac{M^*_{j\cdot l(i)t}}{s^*_{\cdot l(i)}} + \frac{\sum_i v_{ij,t-1}}{\sum_i n_{i,t-1}} - \frac{\sum_l M^*_{j\cdot l,t-1}}{\sum_l s^*_{\cdot l,t-1}}$$

$$\frac{\partial \hat{v}_{ijt,R}}{\partial \hat{v}_{xj,t-1}} = \begin{cases} \left(\frac{(g_2 \sum_{l(i)=l} n_{i,t-1}) - g_1}{g_2^2} \right) \cdot H + \frac{G}{\sum_i n_{i,t-1}} & \text{if } x = i \\ \left(\frac{-g_1}{g_2^2} \right) \cdot H + \frac{G}{\sum_i n_{i,t-1}} & \text{if } l(x) = l(i) \text{ and } x \neq i \\ \frac{G}{\sum_i n_{i,t-1}} & \text{if } l(x) \neq l(i) \end{cases}$$

$$Var(X_{ijta}) = \begin{cases} \sum_{k=1}^{9} s^*_{k1,t-1} q^*_{jk1,t-1}(1 - q^*_{jk1,t-1}) & \text{if } a = 1 \\ \sum_{k=1}^{9} s^*_{k2,t-1} q^*_{jk2,t-1}(1 - q^*_{jk2,t-1}) & \text{if } a = 2 \\ \sum_{k=1}^{9} s^*_{k3,t-1} q^*_{jk3,t-1}(1 - q^*_{jk3,t-1}) & \text{if } a = 3 \\ \sum_{k=1}^{9} s^*_{kl(i)t} q^*_{jkl(i)t}(1 - q^*_{jkl(i)t}) & \text{if } a = 4 \\ n_{a-4,t-1} p_{a-4,j,t-1}(1 - p_{a-4,j,t-1}) & \text{if } a \geq 5 \end{cases}$$

$Cov(X_{ijtm}, X_{ijta})$ is assumed to be 0 when m $\neq$ a because each riding is considered to be independent of all other ridings.

2.4.1 Estimation of Variance

A plug-in estimator is used to estimate the above variances.

Rosenthal's Uniform Estimator

$$\widehat{Var}(\hat{v}_{ijt,U}) = \sum_{a=1}^{A} \left\{ \left(\frac{\partial \hat{v}_{ijt,U}}{\partial X_a} \right)^2 \widehat{Var}(X_a) + \sum_{m<a} 2\widehat{Cov}(X_m, X_a) \right\}$$

$$\widehat{Var}(X_{ijta}) = \begin{cases} \sum_{k=1}^{9} s^*_{k1,t-1} \hat{q}^*_{jk1,t-1}(1 - \hat{q}^*_{jk1,t-1}) & \text{if } a = 1 \\ \sum_{k=1}^{9} s^*_{k2,t-1} \hat{q}^*_{jk2,t-1}(1 - \hat{q}^*_{jk2,t-1}) & \text{if } a = 2 \\ \sum_{k=1}^{9} s^*_{k3,t-1} \hat{q}^*_{jk3,t-1}(1 - \hat{q}^*_{jk3,t-1}) & \text{if } a = 3 \\ \sum_{k=1}^{9} s^*_{kl(i)t} \hat{q}^*_{jkl(i)t}(1 - \hat{q}^*_{jkl(i)t}) & \text{if } a = 4 \\ n_{a-4,t-1} \hat{p}_{a-4,j,t-1}(1 - \hat{p}_{a-4,j,t-1}) & \text{if } a \geq 5 \end{cases}$$

$$\widehat{Cov}(X_{ijtm}, X_{ijta}) = 0$$

Rosenthal's Ratio Estimator

$$\widehat{Var}(\hat{v}_{ijt,R}) = \sum_{a=1}^{A} \left\{ \left(\frac{\partial \hat{v}_{ijt,R}}{\partial X_a} \right)^2 \widehat{Var}(X_a) + \sum_{m<a} 2\widehat{Cov}(X_m, X_a) \right\}$$

$$\widehat{Var}(X_{ijta}) = \begin{cases} \sum_{k=1}^{9} s^*_{k1,t-1}\hat{q}^*_{jk1,t-1}(1-\hat{q}^*_{jk1,t-1}) & \text{if } a=1 \\ \sum_{k=1}^{9} s^*_{k2,t-1}\hat{q}^*_{jk2,t-1}(1-\hat{q}^*_{jk2,t-1}) & \text{if } a=2 \\ \sum_{k=1}^{9} s^*_{k3,t-1}\hat{q}^*_{jk3,t-1}(1-\hat{q}^*_{jk3,t-1}) & \text{if } a=3 \\ \sum_{k=1}^{9} s^*_{kl(i)t}\hat{q}^*_{jkl(i)t}(1-\hat{q}^*_{jkl(i)t}) & \text{if } a=4 \\ n_{a-4,t-1}\hat{p}_{a-4,j,t-1}(1-\hat{p}_{a-4,j,t-1}) & \text{if } a\geq 5 \end{cases}$$

$$\widehat{Cov}(X_{ijtm}, X_{ijta}) = 0$$

2.5 Comparison of Rosenthal's Uniform and Ratio Estimators

Rosenthal proposed two different estimators to estimate the number of votes each party would receive; however, it can be shown that the "ratio voter adjustment", $\hat{v}_{ijt,R}$, is a variant of the "uniform voter-shift adjustment", $\hat{v}_{ijt,U}$. More specifically, the "ratio voter adjustment" assumes that the voting patterns in the ridings are the same for all ridings in region $l(i)$ in the last election. "Uniform voter-shift adjustment" does not make such an assumption.

$$\begin{aligned} \hat{v}_{ijt,U} &= v_{ij,t-1} + \left(\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})\right)\cdot n_{i,t-1} \\ &= v_{ij,t-1} + v_{ij,t-1}\left(\frac{\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})}{v_{ij,t-1}/n_{i,t-1}}\right) \\ &= v_{ij,t-1}\left(1 + \frac{\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r_{ij,t-1}}\right) \end{aligned}$$

Assume that the voting patterns in the ridings are the same for all ridings in region $l(i)$ in the last election, eg. $r_{ij,t-1} = r^*_{jl(i),t-1}$.

$$\begin{aligned} &= v_{ij,t-1}\left(1 + \frac{\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r^*_{jl(i),t-1}}\right) \\ &= v_{ij,t-1}\cdot\frac{r^*_{jl(i),t-1} - r^*_{jl(i),t-1} + \hat{p}^*_{jl(i)t} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r^*_{jl(i),t-1}} \\ &= v_{ij,t-1}\cdot\frac{\hat{p}^*_{jl(i)t} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r^*_{jl(i),t-1}} \\ &= \hat{v}_{ijt,R} \end{aligned}$$

The difference between $\hat{v}_{ijt,R}$ and $\hat{v}_{ijt,U}$ is that $\hat{v}_{ijt,U}$ takes into account the voting pattern of riding i from the last election when estimating the number of votes party j will receive in riding i in the current election. $\hat{v}_{ijt,R}$ makes the assumption that the voting pattern is the same for all ridings in region l in the last election. The "uniform voter-shift adjustment" is expected to perform better than the "ratio voter adjustment" if the riding voting patterns are drastically different than the regional voting pattern in the previous election. Conversely, if the riding voting patterns are similar to the regional voting pattern, the "ratio voter adjustment" may produce similar results as the "uniform voter-shift adjustment" since it is a special case of the "uniform voter-shift adjustment".

2.6 Rationale Behind Rosenthal's Estimators

Since both of the estimators Rosenthal proposed are related, it is sufficient to explain the rationale behind one of them since the same reasoning could be used to explain the other estimator. For simplicity, the rationale behind the "uniform voter-shift adjustment" will be explained.

In the uniform voter-shift adjustment, Rosenthal propose to use the number of votes obtained by party j in riding i in the previous election and modify that by adding or subtracting some number of votes to it.

$$\begin{aligned}\hat{v}_{ijt,U} &= v_{ij,t-1} + \Big(\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})\Big) \cdot n_{i,t-1} \\ &= v_{ij,t-1} + \hat{p}^*_{jl(i)t} \cdot n_{i,t-1} - r^*_{jl(i),t-1} \cdot n_{i,t-1} + (r_{j,t-1} - \hat{p}_{j,t-1}) \cdot n_{i,t-1}\end{aligned}$$

The first term in the equation above, $v_{ij,t-1}$, is the number of votes obtained by party j in riding i in the previous election.

The second term, $\hat{p}^*_{jl(i)t} \cdot n_{i,t-1}$, is an estimate of the number of votes that party j would receive in riding i in the current election under the assumptions that the estimated probability of a voter voting for party j is the same in riding i and in region l, and that the number of voters in riding i is the same as in the previous election.

The third term, $r^*_{jl(i),t-1} \cdot n_{i,t-1}$, is an estimate of the number of votes that party j received in riding i in the previous election under the assumption that the proportion of votes obtained by party j is the same in riding i and in region l.

If the third term, $r^*_{jl(i),t-1} \cdot n_{i,t-1}$, is a good estimate for $v_{ij,t-1}$ and the first three terms are substituted into the equation above, the following can be obtained.

$$\begin{aligned}\hat{v}_{ijt,U} &= v_{ij,t-1} + \hat{v}_{ijt} - \hat{v}_{ij,t-1} + (r_{j,t-1} - \hat{p}_{j,t-1}) \cdot n_{i,t-1} \\ &= \hat{v}_{ijt} + (r_{j,t-1} - \hat{p}_{j,t-1}) \cdot n_{i,t-1}\end{aligned}$$

In other words, the uniform voter-shift adjustment estimator reduces to the sum of an estimate of the number of votes that party j would obtain in riding i in the current election based on the polls and some number of votes based on the overperforming effect.

The overperforming effect could be seen as the difference between an estimate of the number of votes obtained by party j in riding i in the previous election under the assumption that the proportion of votes obtained by party j in riding i is the same as the proportion of votes obtained by party j in Canada, and an estimate of the number of votes obtained by party j in riding i in the previous election under the assumption that the probability of an individual voting for party j in riding i is the same as the probability of an individual voting for party j in Canada.

$$\begin{aligned}\hat{v}_{ijt,U} &= \hat{v}_{ijt} + (r_{j,t-1} - \hat{p}_{j,t-1}) \cdot n_{i,t-1} \\ &= \hat{v}_{ijt} + \hat{v}_{ij,t-1} - \hat{v}_{ij,t-1} \\ &= \hat{v}_{ijt}\end{aligned}$$

Therefore, if the assumptions discussed in the next section are reasonable, Rosenthal's estimators will provide a reasonable estimate for the number of votes that party j would receive in riding i in the current election.

2.7 List of Assumptions Required in Rosenthal's Estimators

In this section, I list the assumptions that are required by both estimators used by Rosenthal.

1. The systematic difference between the population captured by the polls and the population of voters who votes is constant across Canada and is the same as the previous election.
2. The number of voters who votes in each riding is the same as the previous election.

3. The probability of a voter voting for a certain party in a certain riding is the same as the probability of a voter voting for the same party in the region that includes that riding.

2.8 Strength and Weaknesses of Each Estimator

The strengths of Grenier's estimator is that it uses historical information from three previous elections instead of only one previous election. Grenier also incorporates candidate level variables in his estimator. Most importantly, it only requires one of the three assumptions listed in section 2.7. More specifically, Grenier's estimator only requires the voting pattern to be the same for all ridings in a region. The weakness of Grenier's estimator is that it does not take into account the possibility that there may be a systematic difference between the population who participated in the polls and the population that votes on election day.

The strength of Rosenthal's estimators is that they account for the possibility of a systematic difference between the population who participated in the polls and the population that votes on election day. The weaknesses of Rosenthal's estimators are that they only use polling data and election results from the previous election. Also, Rosenthal did not incorporate candidate-level information in both of his estimators. Lastly, Rosenthal's estimators require assumptions that are likely to be violated in real elections.

Chapter 3

Numerical Study

3.1 Motivation

The assumptions required by Rosenthal's estimators as listed in Section 2.7 may not be appropriate in real elections. In this section, data from past Canadian federal elections[1][2][3][4] and their corresponding polling results[8][9][10][11] are used to show that these assumptions are likely to be violated in a real election setting.

3.1.1 Assumption 1: Uniform Overperforming Effect and the Same as the Previous Election

Rosenthal suspects that the sample of individuals who were polled is not representative of the individuals who will vote on election day. In order to estimate the difference in the probabilities of choosing party j in these two groups of people, Rosenthal assumed that this difference is constant in every region in Canada and is the same as the previous election.

Table 3.1: National Overperforming Effect Observed in 2008, 2011, and 2015 Election. Overperforming effects for 2008 Election was obtained from Rosenthal, J.S. (2011). Was the conservative majority predictable? *The Canadian Journal of Statistics*, 39(4):721-733.

	2015	2011	2008
Conservative	0.5	4.1	3.0
Liberals	1.2	-0.8	-0.2
NDP	-1.2	-1.2	-1.2
Bloc	0.6	0.3	0.1
Green	-1.2	-2.0	-2.4
Others	0.2	-0.4	0.6

Table 3.2: Regional Overperforming Effect Observed in the 2015 Canadian Federal Election

	Canada	Atlantic	Quebec	Ontario	Prairies and Territories	British Columbia
Conservative	0.7	-0.9	-2.8	2.5	3.5	-1.8
Liberals	2.3	5.1	6.2	0.4	-0.3	2.6
NDP	-1.7	-3.9	-1.2	-2.1	-3.2	-1
Bloc	-0.3	0	-1.5	0	0	0
Green	-0.9	-0.7	-0.6	-1.1	-0.9	-0.4
Others	-0.1	0.3	-0.1	0.3	1.0	0.5

Table 3.1 shows that overperforming effect is not constant between consecutive elections. In fact, the overperforming effect changes a lot for the Conservatives and Liberals in these three elections. Table 3.2 shows that overperforming effect is not the same in different regions in Canada. The overperforming effect varies drastically from one region to the next. The overperforming effect changes a lot even if two regions are next to one another. For example, the overperforming effect for the Conservatives in Quebec is -2.8, while the overperforming effect for the Conservatives in Ontario is 2.5. The difference in overperforming effect is more than 5 percentage points even though these two regions are neighbours. Therefore, tables 3.1 and 3.2 show that Rosenthal's assumption about the overperforming effects are inappropriate for Canadian federal elections.

3.1.2 Assumption 2: Number of Voters

The estimators proposed by Rosenthal require the assumption that the number of voters voting in a certain riding is the same as the previous election. Based on figure 3.1, it can be seen that this assumption is inappropriate. Each point in this figure represents one riding. The diagonal line is the $y = x$ line. If a point falls on this line, it means that the number of voters is equal in two consecutive elections in that riding. However, it can be seen that there are fewer voters in the 2008 election compared to the 2006 election. Also, it can be seen that there are more voters in the 2011 election compared to the 2008 election. Lastly, there is a significant increase in the number of voters in the 2015 election compared to the 2011 election. Based on figure 3.1, we can conclude that Rosenthal's assumption about the number of voters being the same as that of the previous election is inappropriate for Canadian federal elections.

3.1.3 Assumption 3: Uniform Probability of Voting for a Certain Party

Both of the estimators proposed by Rosenthal and Grenier require the assumption that the probability of an individual voting for party j is the same for all ridings within the

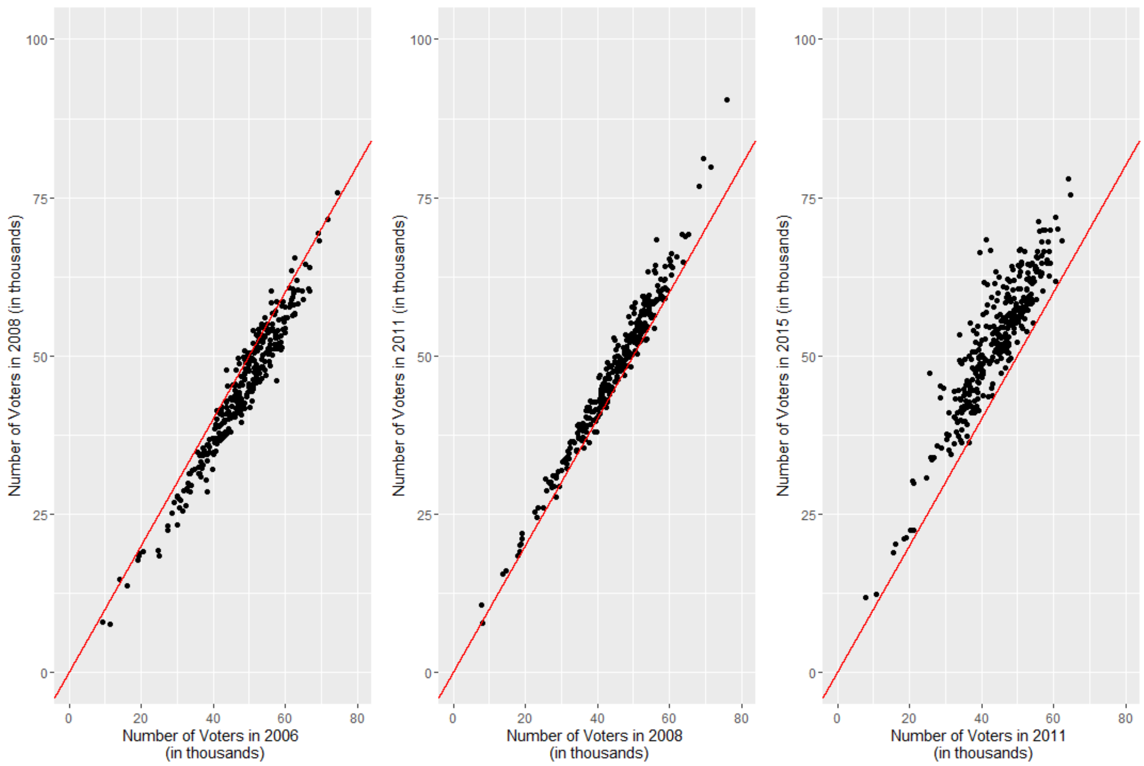

Figure 3.1: Number of Voters Between Consecutive Elections by Riding

same region. Based on figures 3.2a and 3.2b, it can be seen that this assumption is clearly inappropriate for the 2011 and 2015 elections. In all regions in each of the elections, it is clear that the proportion of votes obtained by any party is not constant. For example, the proportion of votes received by the Liberals in British Columbia ranges from about 10% to about 60% in the 2015 Canadian federal election. Therefore, the regional mean of the probability of an individual voting for party j is a bad estimate of the probability of an individual voting for party j in all ridings belonging to that region. Based on figures 3.2a and 3.2b, it is clear that the regional proportion is different from the proportions at a riding level. In other words, $p_{ijt} \neq p^*_{jl(i)t}$ for all $i = 1, \ldots, I(t)$ and $j = 1, \ldots, J$.

3.2 Assumptions

In this numerical study, all combinations of the three assumptions listed in section 2.7 are violated to various degrees. To quantify the levels of departure from each assumption, the parameters δ, γ, and ϕ represent the departure from a specific assumption, which will be explained below. If a parameter is set to 0, it represents the corresponding assumption is not violated and that it is true in that simulation setting. Below is a list of the assumptions and the parameter that corresponds to each assumption.

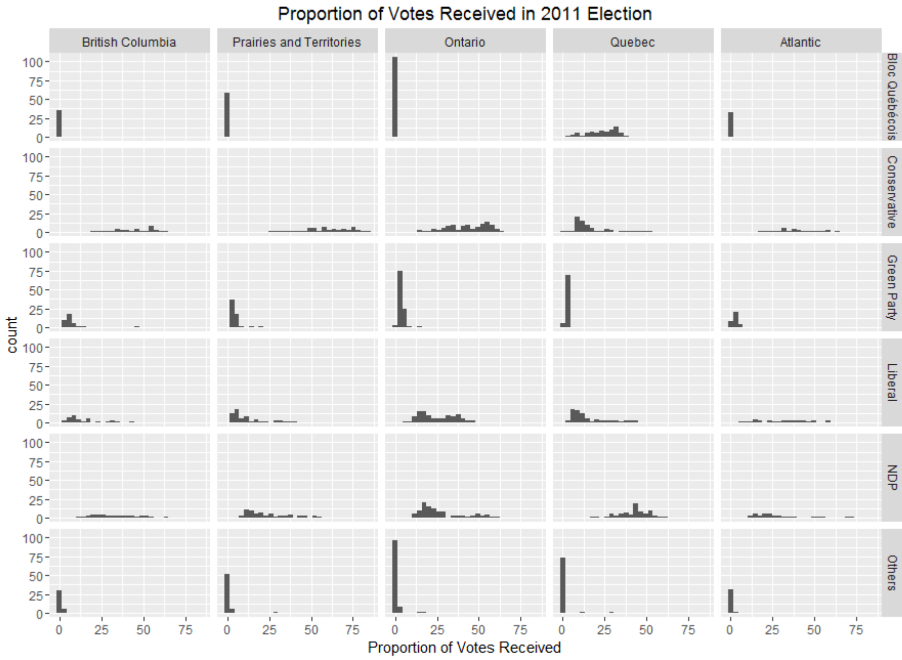

Figure 3.2a: Proportion of Votes Received by Each Party in Different Regions in the 2011 Election

1. The systematic difference between the population captured by the polls and the population of voters who votes is constant across Canada and is constant between elections. The parameter δ is used to represent the level of departure from assumption 1. In this simulation study, δ takes the value of 0, 0.5, 1, 2, or 5. If δ is equal to 5, that means the difference in probability of voting for a certain party between the population who participate in the polls and the population who votes will deviate by up to 12.5%.

2. The number of voters who votes in each riding is constant between consecutive elections. The parameter ϕ is used to represent the level of departure from assumption 2. In this simulation study, δ takes the value of -0.3, -0.05, 0, 0.05, or 0.3. The number of voters will change by $\phi \cdot 100\%$ in the second election compared to the first election.

3. The probability of a voter voting for a certain party in a certain riding is the same as the probability of a voter voting for the same party in the region that includes that riding. The parameter γ is used to represent the level of departure from assumption 3. In this simulation study, γ takes the value of -10, -1, 0, 1, or 10. If γ is equal to 10, that means the difference in probability of voting for a certain party in a region will deviate by up to 25%.

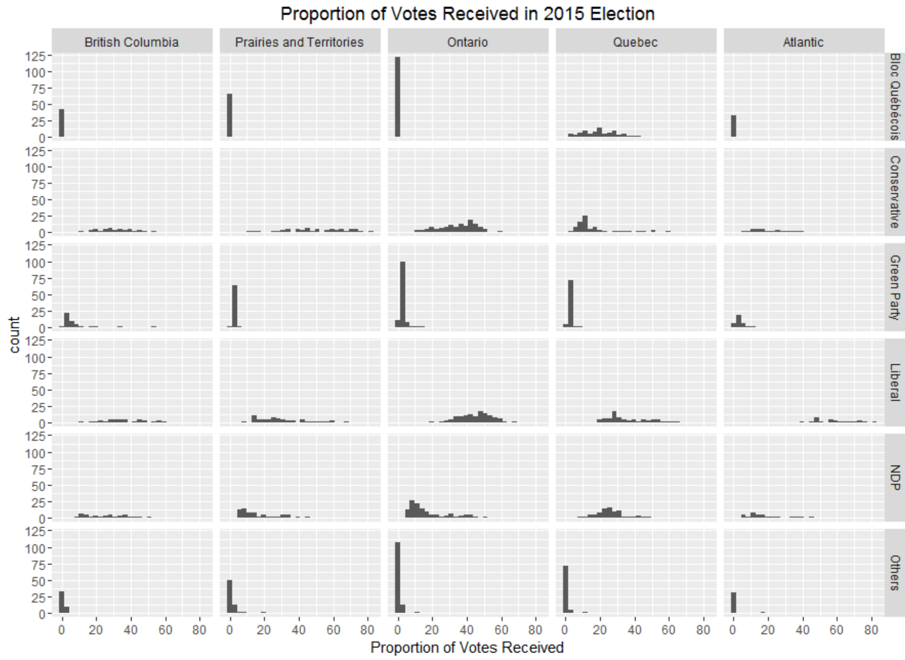

Figure 3.2b: Proportion of Votes Received by Each Party in Different Regions in the 2015 Election

3.3 Design

3.3.1 Description of Simulation Settings

In this simulation study, I used a hypothetical country as the setting. In this country, there are three different regions. In region 1, there are 10 ridings. In region 2, there are 40 ridings. In region 3, there are 20 ridings. In this country, there are a total of three parties in each election; however, party 3 only competes in region 3. A diagram representation of this hypothetical country can be found in figure 3.3.

Parameters Used in the Simulation

- The probability of an individual voting for party j in region l for election s is

$$p^*_{jls} \quad \forall s = t, t-1$$

  The list of p^*_{jls} used in the simulation study can be found in Table 3.3.

- The probability of an individual voting for party j in riding i in election s is

$$p_{ijs} = p^*_{jl(i)s} + \beta_{ijs}\gamma \quad \forall s = t, t-1$$

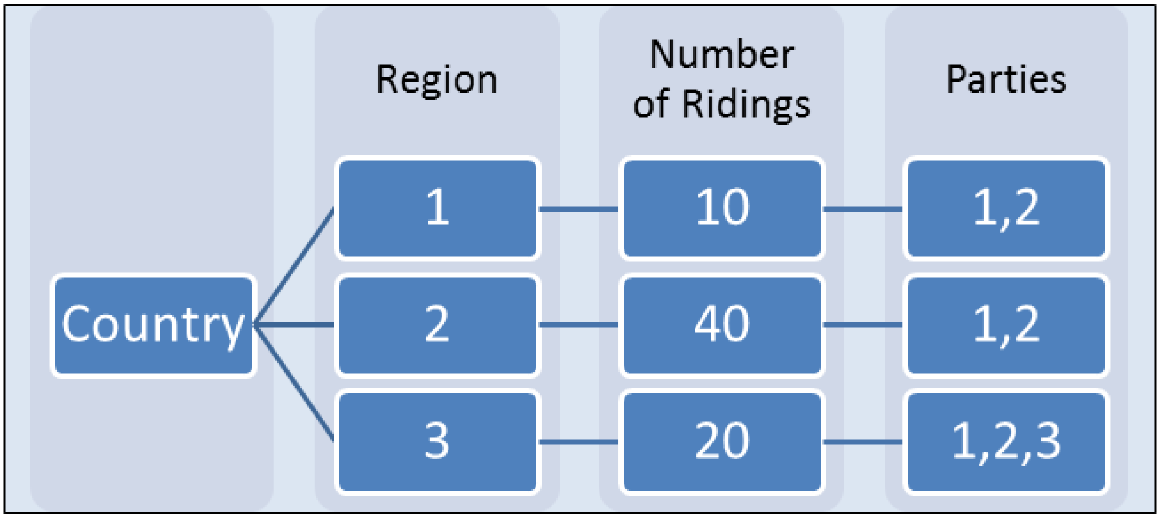

Figure 3.3: Diagram representation of the hypothetical country

Table 3.3: List of p^*_{jls} used in the simulation study

p^*_{jlt}

l	$j=1$	$j=2$	$j=3$
1	0.40	0.60	0
2	0.65	0.35	0
3	0.25	0.40	0.35

$p^*_{jl,t-1}$

l	$j=1$	$j=2$	$j=3$
1	0.50	0.50	0
2	0.70	0.30	0
3	0.30	0.50	0.20

where,

$$\beta_{i1s} \sim N(0, 0.01^2) \quad \forall i = 1, \ldots, 70; s = t, t-1$$
$$\beta_{i2s} \sim N(0, 0.01^2) \quad \forall i = 51, \ldots, 70; s = t, t-1$$
$$\beta_{i3s} \equiv 0 \quad \forall i = 1, \ldots, 50; s = t, t-1$$

subject to,

$$\beta_{i1s} \neq 0 \quad \forall i = 1, \ldots, 70; s = t, t-1$$
$$\beta_{i2s} \neq 0 \quad \forall i = 1, \ldots, 70; s = t, t-1$$
$$\beta_{i3s} \neq 0 \quad \forall i = 51, \ldots, 70; s = t, t-1$$
$$\sum_j \beta_{ijs} = 0 \quad \forall i = 1, \ldots, 70; s = t, t-1$$
$$\sum_{l(i)=l} \beta_{ijs} = 0 \quad \forall l = 1, 2, 3; s = t, t-1$$

Refer to Appendix B and C for the list of p_{ijs} used in this simulation study.

- The difference between the probabilities of an individual voting for party j in region l in the election and in the polls is

$$\Delta^*_{jls} \quad \forall s = t, t-1$$

The list of Δ^*_{jls} used in the simulation study can be found in Table 3.4.

Table 3.4: List of Δ^*_{jls} used in the simulation study

Δ^*_{jlt}

l \ j	1	2	3
1	-0.015	0.015	0
2	-0.020	0.020	0
3	-0.025	0.010	0.015

$\Delta^*_{jl,t-1}$

l \ j	1	2	3
1	-0.025	0.025	0
2	-0.025	0.025	0
3	-0.015	0.010	0.005

- The probability of an individual from region l voting for party j in the poll for election s is

$$q^*_{jls} = p^*_{jls} + \Delta^*_{jls} \quad \forall j = 1, \ldots, J; l = 1, \ldots, L; s = t, t-1$$

- The total number of voters in riding i in election s is

$$n_{is} \quad \forall s = t, t-1$$

$$n_{i,t-1} = 10000(1+\phi) \quad \forall i = 1, \ldots, 70$$

$$n_{it} = 10000(1+\phi)^2 \quad \forall i = 1, \ldots, 70$$

3.3.2 Choosing Parameters

The following settings are chosen to examine all possible combinations of the effect of departure from the various assumptions. In setting 1, the ideal case, that is the case that all three assumptions hold, is considered to provide a baseline comparison to all other settings. In settings 2 to 4, only one of the three assumption is violated. The effect of the departure from each of the three assumptions on the estimators can be examined from the results of these settings. Settings 5 to 7 allow the examination of the departure from two of the three assumptions on the estimators. The interaction between the assumptions can be examined based on settings 5 to 7. Lastly, the effect of the departure from all three assumptions on the estimators is considered in setting 8.

Setting 1: All of assumptions 1, 2, and 3 hold

1. γ is set to 0.
2. δ is set to 0.
3. ϕ is set to 0.

Setting 2: Assumption 1 violated, assumptions 2 and 3 hold

1. γ is set to 0.
2. δ is set to four different levels to reflect the level of departure from assumption 1. More specifically, δ is set to 0.5, 1, 2, and 5 to generate different overperforming effects.
3. ϕ is set to 0.

Setting 3: Assumption 2 violated, assumptions 1 and 3 hold

1. γ is set to 0.
2. δ is set to 0.
3. ϕ is set to four different levels to reflect the degree of departure from assumption 2. More specifically, ϕ is set to -0.3, -0.05, 0.05, and 0.3.

Setting 4: Assumption 3 violated, assumptions 1 and 2 hold

1. γ is set to four different levels to reflect the departure from assumption 3. More specifically, γ is set to -10, -1, 1, and 10.
2. δ is set to 0.
3. ϕ is set to 0.

Setting 5: Assumptions 2 and 3 violated, assumption 1 holds

1. γ is set to four different levels to reflect the departure from assumption 3. More specifically, γ is set to -10, -1, 1, and 10.
2. δ is set to 0.

3. ϕ is set to four different levels to reflect the degree of departure from assumption 2. More specifically, ϕ is set to -0.3, -0.05, 0.05, and 0.3.

Setting 6: Assumptions 1 and 3 violated, assumption 2 holds

1. γ is set to four different levels to reflect the departure from assumption 3. More specifically, γ is set to -10, -1, 1, and 10.
2. δ is set to four different levels to reflect the level of departure from assumption 1. More specifically, δ is set to 0.5, 1, 2, and 5 to generate different overperforming effects.
3. ϕ is set to 0.

Setting 7: Assumptions 1 and 2 violated, assumption 3 holds

1. γ is set to 0.
2. δ is set to four different levels to reflect the level of departure from assumption 1. More specifically, δ is set to 0.5, 1, 2, and 5 to generate different overperforming effects.
3. ϕ is set to four different levels to reflect the degree of departure from assumption 2. More specifically, ϕ is set to -0.3, -0.05, 0.05, and 0.3.

Setting 8: All of assumptions 1, 2, and 3 violated

1. γ is set to four different levels to reflect the departure from assumption 3. More specifically, γ is set to -10, -1, 1, and 10.
2. δ is set to four different levels to reflect the level of departure from assumption 1. More specifically, δ is set to 0.5, 1, 2, and 5 to generate different overperforming effects.
3. ϕ is set to four different levels to reflect the degree of departure from assumption 2. More specifically, ϕ is set to -0.3, -0.05, 0.05, and 0.3.

3.3.3 Simulation Procedures

Voting Results

1. Generate $(v_{i1s}, v_{i2s}, v_{i3s})$ which follows a multinomial distribution with n_{is} trials and probabilities $(p_{i1s}, p_{i2s}, p_{i3s})$ for all $i = 1, \ldots, 70$ and $s = t, t-1$.

2. Calculate r_{ijs}

$$r_{ijs} = \frac{v_{ijs}}{n_{is}} \quad \forall s = t, t-1$$

3. Calculate r^*_{jls}

$$r^*_{jls} = \frac{\sum_{l(i)=l} v_{ijt}}{n^*_{lt}} \quad \forall s = t, t-1$$

4. Calculate $r_{j,t-1}$

$$r_{j,t-1} = \frac{\sum_i v_{ij,t-1}}{n_{t-1}} \quad \forall j = 1,2,3$$

Polling Results

1. Set $m_1 = 500$, $m_2 = 2000$, and $m_3 = 1000$.
2. Generate $(M^*_{1ls}, M^*_{2ls}, M^*_{3ls})$ which follows a multinomial distribution with m_l trials and probabilities $(q^*_{1ls}, q^*_{2ls}, q^*_{3ls})$ for all $l = 1,2,3$ and $s = t, t-1$.
3. Calculate $\hat{p}^*_{jlt}$

$$\hat{p}^*_{jlt} = \frac{M^*_{jlt}}{m_l} \quad \forall j = 1,2,3; l = 1,2,3$$

4. Calculate $\hat{p}_{j,t-1}$

$$\hat{p}_{j,t-1} = \frac{\sum_l M_{jl,t-1}}{\sum_l m_l} \quad \forall j = 1,2,3$$

Obtaining Estimates

1. Predict p_{ijt} by using Rosenthal's Uniform and Ratio Estimator
   - Rosenthal's Uniform Estimator

$$\hat{v}_{ijt,U} = v_{ij,t-1} + \left(\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})\right) \cdot n_{i,t-1}$$

   - Rosenthal's Ratio Estimator

$$\hat{v}_{ijt,R} = v_{ij,t-1} \frac{\hat{p}^*_{jl(i)t} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r^*_{jl(i),t-1}}$$

   - Rosenthal's estimators predicts v_{ijt}, obtain $\hat{p}_{ijt} = \frac{\hat{v}_{ijt}}{n_{it}}$.
   - Estimate the variances for the two estimators using the equations found in section 2.4.1.

- Obtain the percent of correct calls from each iteration by comparing the actual winner from each riding against the predicted winner from each riding.

$$\psi_i = \begin{cases} 1 & \text{if } c_{ijt} = \hat{c}_{ijt} \quad \forall j \\ 0 & \text{Otherwise} \end{cases}$$

  The percent of correct calls can be obtained by summing ψ_i over all i and dividing by the number of ridings, I.

- Obtain the estimated number of seats from the list of predicted winners. The estimated seats for party j can be obtained by summing $\hat{c}_{ijt}$ over all i.

2. Repeat the simulation procedures 1,000 times, saving the results each time

3.4 Results

In this section, each figure contains four sets of plots. In each plot, the red line represents the true value and the blue line represents the average of the estimates from the simulation. The blue shaded regions represent the 2.5 to 97.5 percentile of the estimated value based on the simulation results.

In each set of plots, the results are plotted separately for the p_{ijt} or v_{ijt} and the standard error and separated by parties. In each set of plots, the 2.5 to 97.5 percentile of the estimated p_{ijt} or v_{ijt} is plotted against the true value for each riding in the top 3 plots, and the 2.5 to 97.5 percentile of the estimated standard error for p_{ijt} or v_{ijt} is plotted against the "true standard error" in the bottom 3 plots. The "true standard error" is obtained by substituting the true parameters into equations 14 and 15 in section 2.4 and taking the square root of the result divided by 1000.

The two sets of plots on the left show the results in terms of the estimated probability of a voter voting for the specific party for both Rosenthal's uniform and ratio estimator. The two sets of plots on the right show the results in terms of the estimated number of votes for both Rosenthal's uniform and ratio estimator. In both instances, results from Rosenthal's uniform estimator will be displayed on the top set of plots, and the results from Rosenthal's ratio estimator will be displayed on the bottom set of plots.

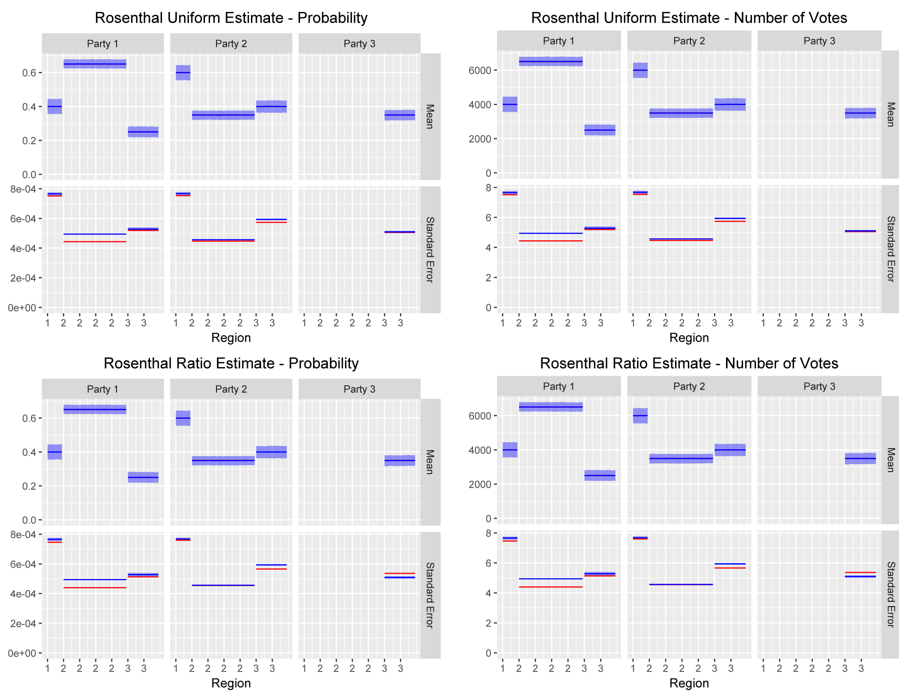

Figure 3.4: Simulation results of Setting 1 ($\gamma = 0, \delta = 0, \phi = 0$)

Setting 1: All of Assumptions 1, 2, and 3 hold

Estimating probability of votes

Under the ideal situation (see figure 3.4), it can be observed that both estimators perform reasonably well. Both the uniform and ratio estimators produce decent estimates for p_{ijt}. The true standard errors from the uniform estimator for p_{ijt} are contained in the 2.5 to 97.5 percentile except for party 3. The estimated standard error is not good for the ratio estimator. Although the results are similar, the uniform estimator (MSE: 0.0429) performs slightly better than the ratio estimator (MSE: 0.0434).

Estimating number of votes

Under the ideal situation (see figure 3.4), it can be observed that both the uniform and ratio estimators produce decent estimates for v_{ijt}. The true standard errors from the uniform estimator for v_{ijt} are contained in the 2.5 to 97.5 percentile except for party 3. The estimated standard error is not good for the ratio estimator. Although the results are similar, the uniform estimator (MSE: 4314747) performs slightly better than the ratio estimator (MSE: 4369298).

Setting 2: Assumption 1 violated, Assumptions 2 and 3 hold

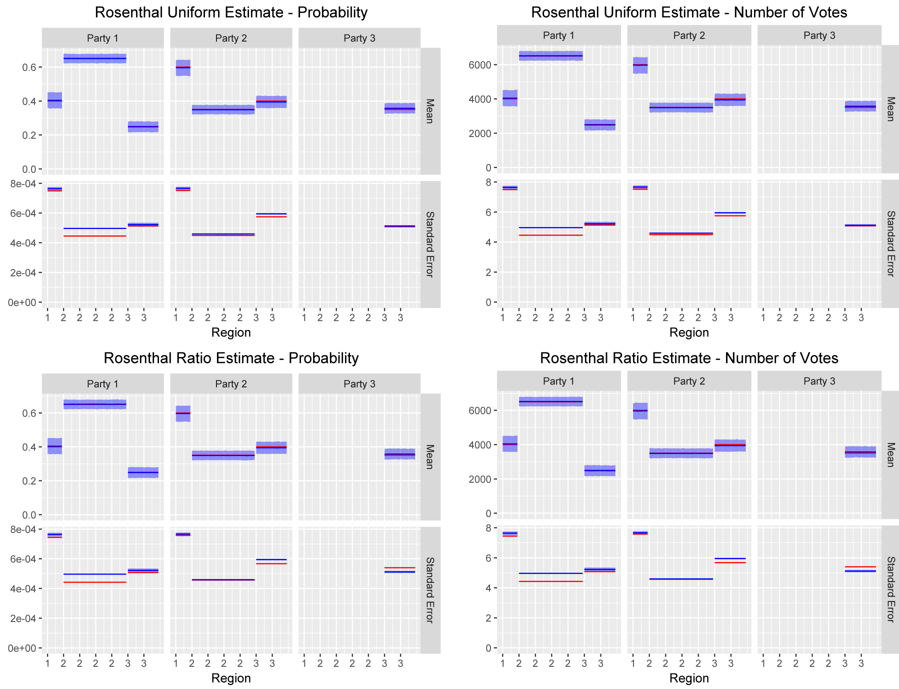

Figure 3.5a: Simulation results of Setting 2 ($\gamma = 0, \delta = 0.5, \phi = 0$)

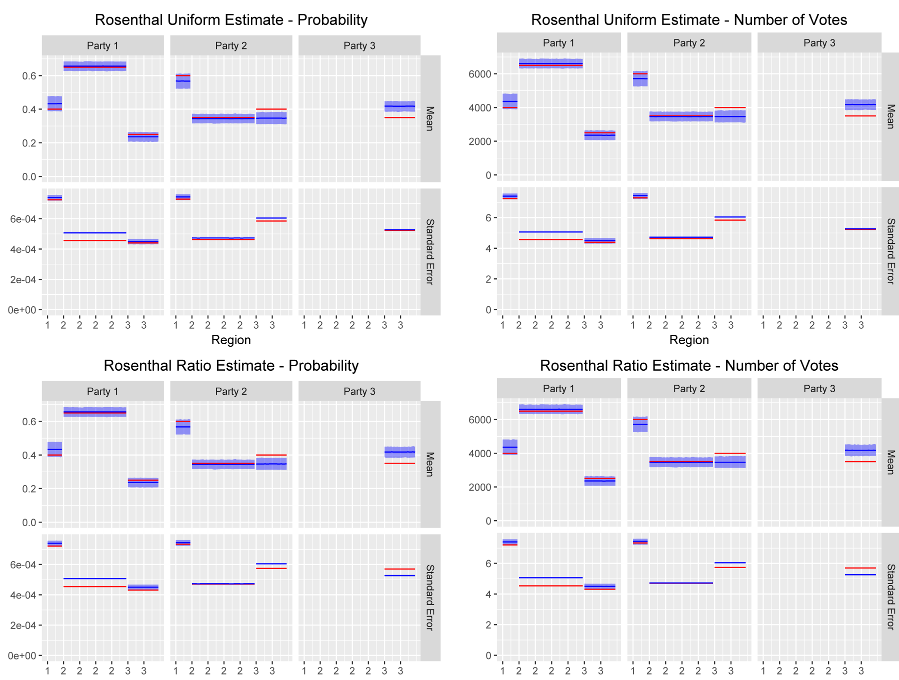

Figure 3.5b: Simulation results of Setting 2 ($\gamma = 0, \delta = 5, \phi = 0$)

Estimating probability of votes

Under the situation where there is a slight difference between the population who participates in the polls and the population who votes on election day (see figure 3.5a), it can be observed that both estimators produce similar results. Both estimators produce a decent estimate for p_{ijt}. The estimated standard error for p_{ijt} are better for the uniform estimator than the ratio estimator, but the 2.5 to 97.5 percentile of the estimated standard error does not contain the true standard error most of the time. The uniform estimator (MSE: 0.0472) performs slightly better than the ratio estimator (MSE: 0.0476).

If there is a large difference between the population who participates in the polls and the population who votes on election day (see figure 3.5b), it can be observed that both estimators produce similar estimates for p_{ijt}. Neither the estimated p_{ijt} nor the estimated standard error are good. The uniform estimator (MSE: 0.2219) performs slightly better than the ratio estimator (MSE: 0.2223).

Estimating number of votes

Under the situation where there is a slight difference between the population who participates in the polls and the population who votes on election day (see figure 3.5a), it can be observed that both estimators produce a decent estimate for v_{ijt}. The estimated standard error for v_{ijt} are better for the uniform estimator than the ratio estimator, but the 2.5 to 97.5 percentile of the estimated standard error does not contain the true standard error most of the time. The uniform estimator (MSE: 4755183) performs slightly better than the ratio estimator (MSE: 4812267).

If there is a large difference between the population who participates in the polls and the population who votes on election day (see figure 3.5b), it can be observed that both estimators produce similar estimates for v_{ijt}. Neither the estimated v_{ijt} nor the estimated standard error are good. The uniform estimator (MSE: 22435602) performs better than the ratio estimator (MSE: 22518411).

Setting 3: Assumption 2 violated, Assumptions 1 and 3 hold

Estimating probability of votes

If there is a small difference in the number of voters between the two elections (see figure 3.6a), the estimated p_{ijt} from both estimators is good. The estimated standard error is

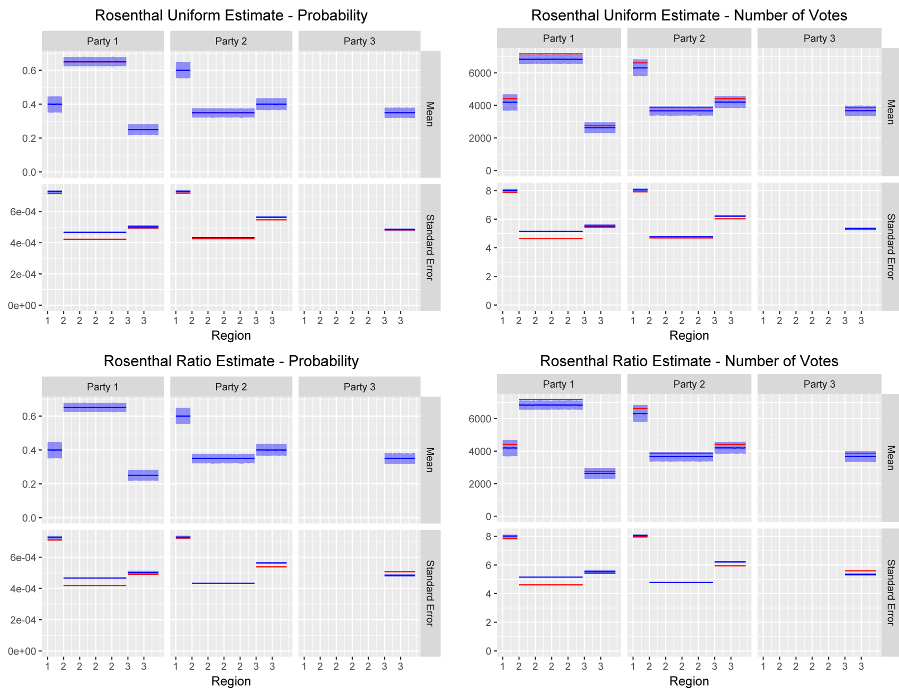

Figure 3.6a: Simulation results of Setting 3 ($\gamma = 0, \delta = 0, \phi = 0.05$)

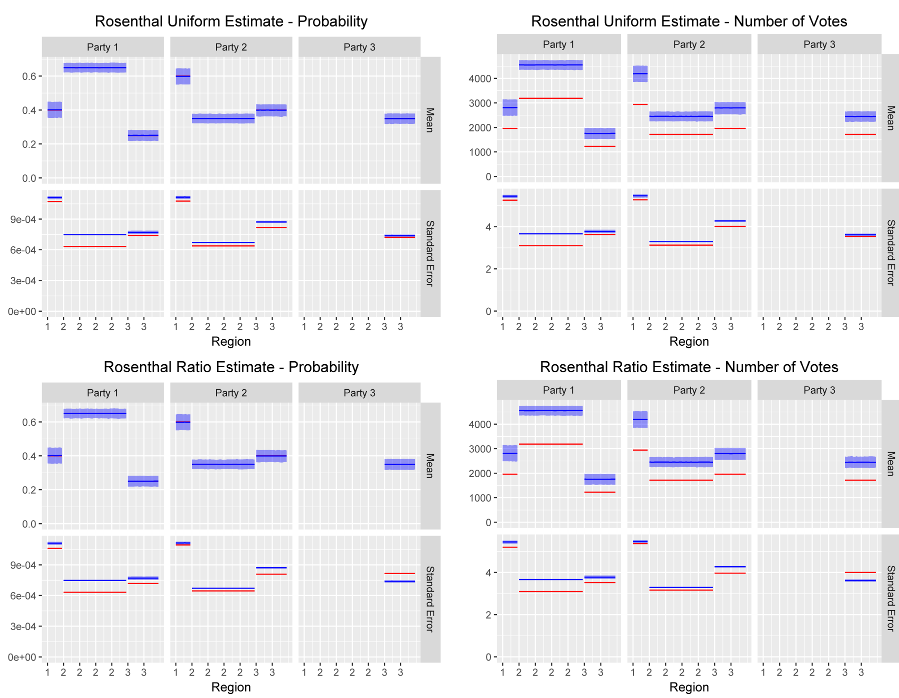

Figure 3.6b: Simulation results of Setting 3 ($\gamma = 0, \delta = 0, \phi = -0.3$)

better for the uniform estimator than the ratio estimator. The uniform estimator (MSE: 0.0441) performs slightly better than the ratio estimator (MSE: 0.0445).

If there is a large difference in the number of voters between the two elections (see figure 3.6b), the estimated p_{ijt} is bad for both estimators. Also, the estimated standard error is bad for both estimators. The uniform estimator (MSE: 0.0453) performs better than the ratio estimator (MSE: 0.0460).

Estimating number of votes

If there is a small difference in the number of voters between the two elections (see figure 3.6a), the true v_{ijt} is barely contained in the 2.5 to 97.5 percentile. The estimated standard error is better for the uniform estimator than the ratio estimator. The uniform estimator (MSE: 14164935) performs slightly better than the ratio estimator (MSE: 14220814).

If there is a large difference in the number of voters between the two elections (see figure 3.6b), the estimated v_{ijt} is bad for both estimators. Also, the estimated standard error is bad for both estimators. The uniform estimator (MSE: 151502127) performs better than the ratio estimator (MSE: 151540365).

Setting 4: Assumption 3 violated, Assumptions 1 and 2 hold

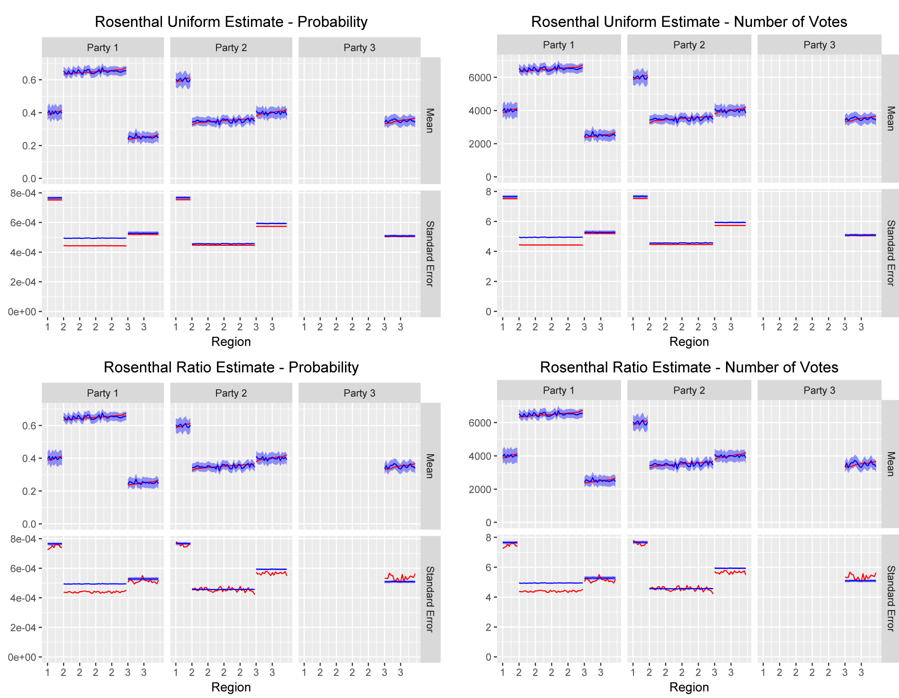

Figure 3.7a: Simulation results of Setting 4 ($\gamma = 1, \delta = 0, \phi = 0$)

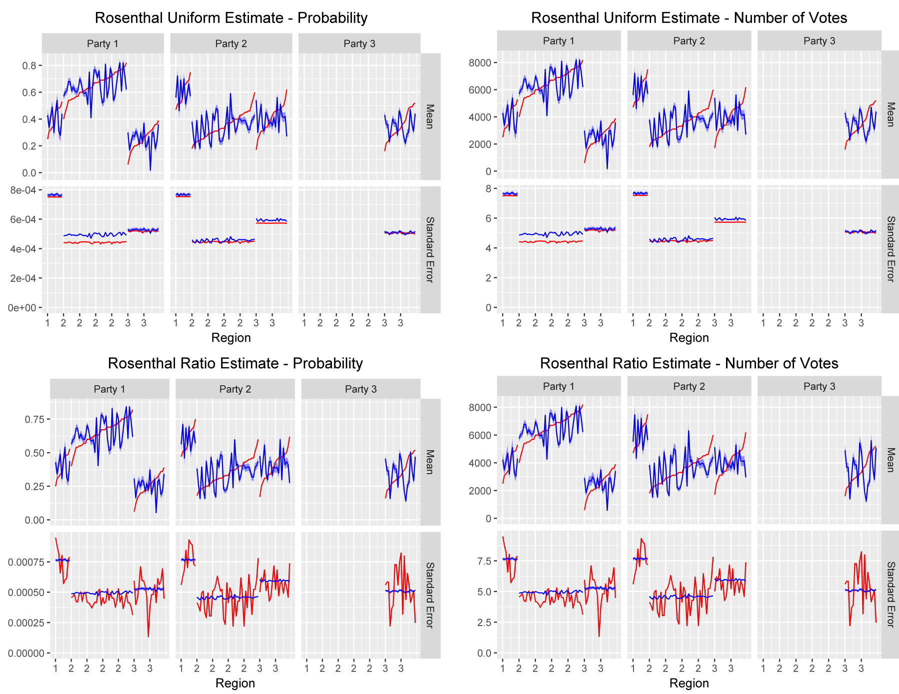

Figure 3.7b: Simulation results of Setting 4 ($\gamma = -10, \delta = 0, \phi = 0$)

Estimating probability of votes

If the probability of a voter voting for a certain party in a certain riding deviates slightly from the probability of a voter voting for the same party in the region (see figure 3.7a), it can be observed that the estimator for p_{ijt} is reasonable for both estimators. The estimated standard error of p_{ijt} for the uniform estimator are good except in region 3, but they are not reliable for the ratio estimator. In this scenario, the uniform estimator (MSE: 0.0665) performs better than the ratio estimator (MSE: 0.0690).

If the probability of a voter voting for a certain party in a certain riding deviates significantly from the probability of a voter voting for the same party in the region (see figure 3.7b), it can be observed that neither of the estimators provide a reasonable estimate for p_{ijt}. The estimated standard error for p_{ijt} are not good for both the uniform and ratio estimators. The uniform estimator (MSE: 2.6048) performs better than the ratio estimator (MSE: 2.8610).

Estimating number of votes

If the probability of a voter voting for a certain party in a certain riding deviates slightly from the probability of a voter voting for the same party in the region (see figure 3.7a), it can be observed that the estimator for v_{ijt} is reasonable for both estimators. The estimated standard error of v_{ijt} for the uniform estimator are good except in region 3, but they are not reliable for the ratio estimator. In this scenario, the uniform estimator (MSE: 6668882) performs better than the ratio estimator (MSE: 6975210).

If the probability of a voter voting for a certain party in a certain riding deviates significantly from the probability of a voter voting for the same party in the region (see figure 3.7b), it can be observed that neither of the estimators provide a reasonable estimate for v_{ijt}. The estimated standard error for v_{ijt} are not good for both the uniform and ratio estimators. The uniform estimator (MSE: 260494151) performs better than the ratio estimator (MSE: 271985421).

Setting 5: Assumptions 2 and 3 violated, Assumption 1 holds

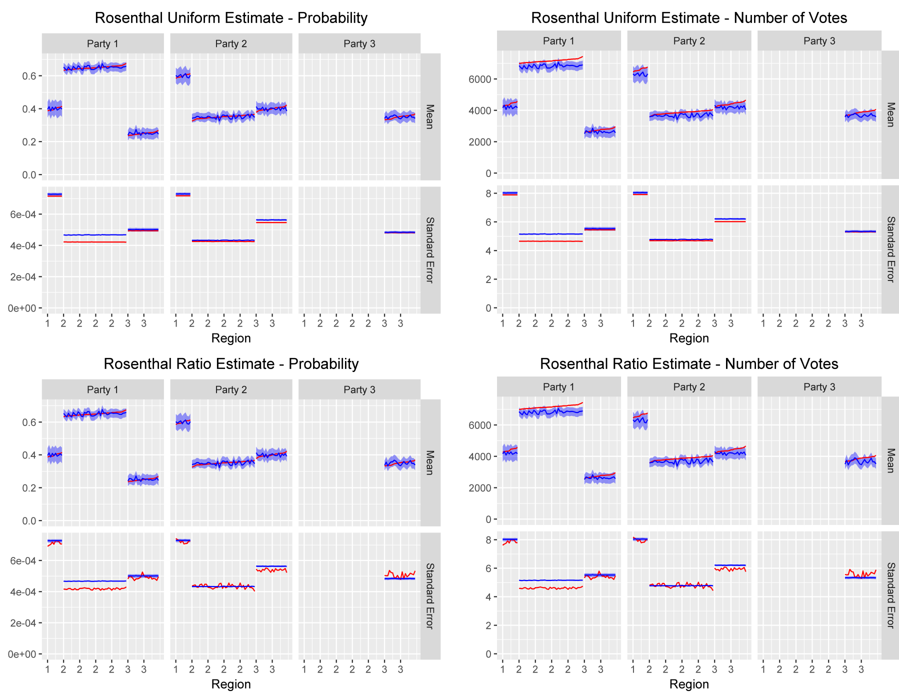

Figure 3.8a: Simulation results of Setting 5 ($\gamma = 1, \delta = 0, \phi = 0.05$)

Estimating probability of votes

If the number of voters between the two elections are slightly different and the riding probability is slightly different than the regional probability (see figure 3.8a), both estimators provide decent estimates for p_{ijt}. The estimated standard error is better for the uniform estimator than the ratio estimator. The uniform estimator (MSE: 0.0675) performs slightly better than the ratio estimator (MSE: 0.0700).

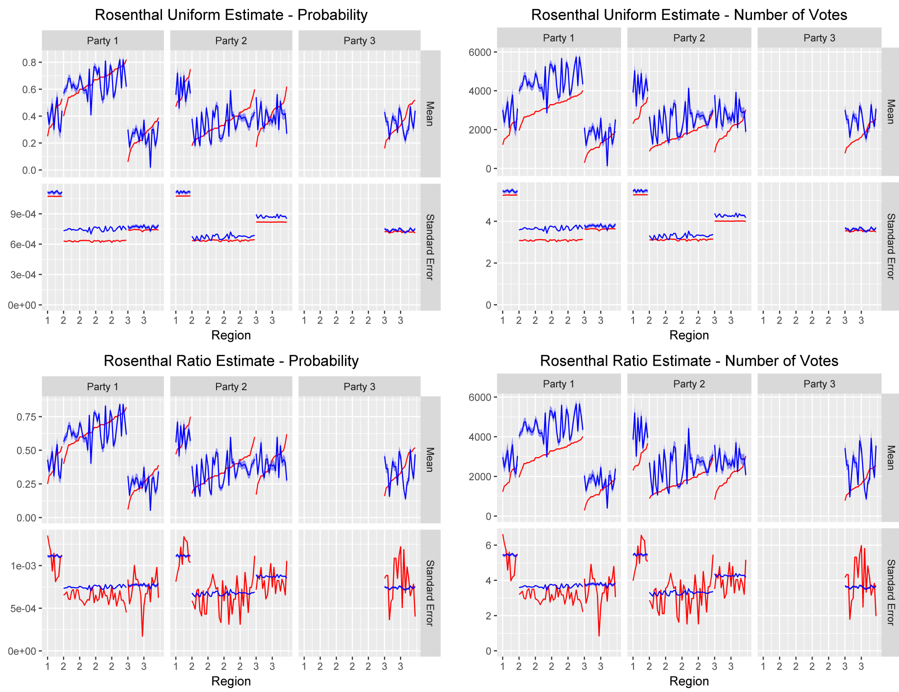

Figure 3.8b: Simulation results of Setting 5 ($\gamma = -10, \delta = 0, \phi = -0.3$)

If the number of voters between the two elections are significantly different and the riding probability is significantly different than the regional probability (see figure 3.8b), neither estimators provide good estimates for p_{ijt}. The estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator, but neither captures the real variance. The uniform estimator (MSE: 2.6067) performs better than the ratio estimator (MSE: 2.8640).

Estimating number of votes

If the number of voters between the two elections are slightly different and the riding probability is slightly different than the regional probability (see figure 3.8a), neither estimators provide a good estimate for v_{ijt}. The estimated standard error is better for the uniform estimator than the ratio estimator. The uniform estimator (MSE: 16981437) performs slightly better than the ratio estimator (MSE: 17316510).

If the number of voters between the two elections are significantly different and the riding probability is significantly different than the regional probability (see figure 3.8b), neither estimators provide good estimates for v_{ijt}. The estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator, but neither captures the real variance. The uniform estimator (MSE: 241625887) performs better than the ratio estimator (MSE: 248802151).

Setting 6: Assumptions 1 and 3 violated, Assumption 2 holds

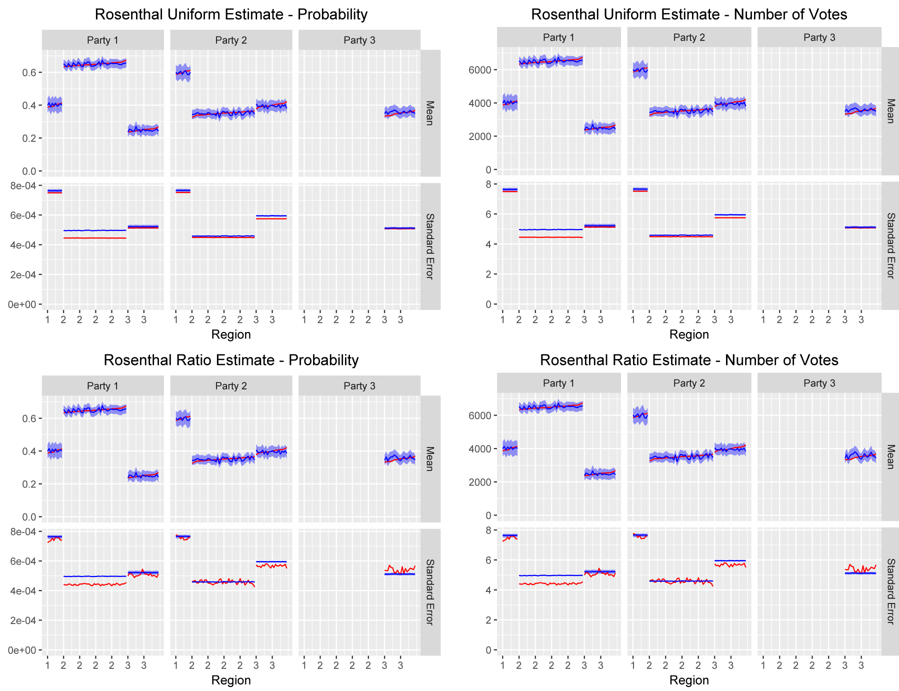

Figure 3.9a: Simulation results of Setting 6 ($\gamma = 1, \delta = 0.5, \phi = 0$)

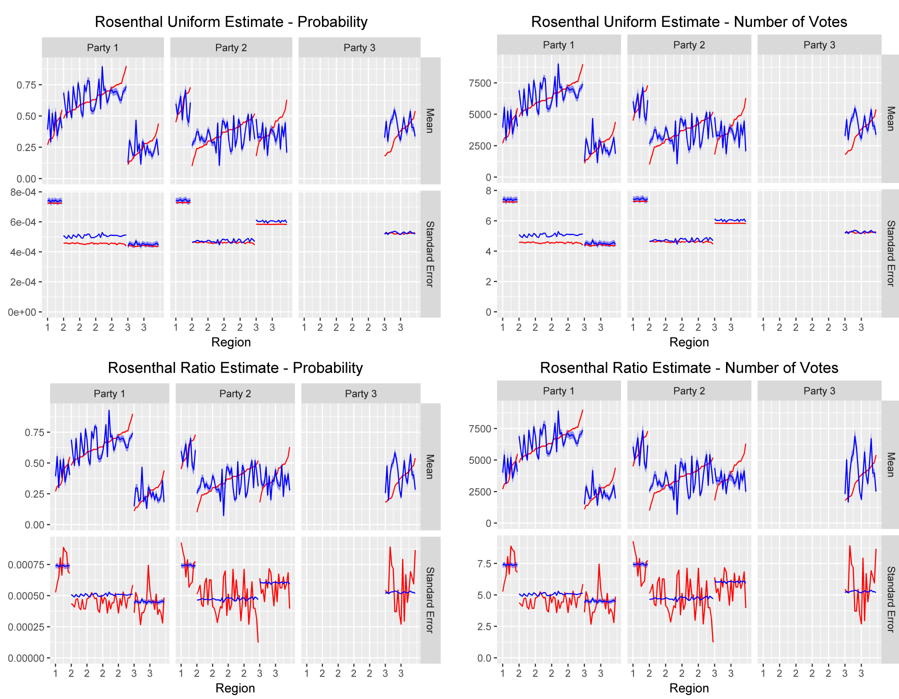

Figure 3.9b: Simulation results of Setting 6 ($\gamma = 10, \delta = 5, \phi = 0$)

Estimating probability of votes

If the polling population and the voting population are slightly different and the riding probability deviates slightly from the regional probability (see figure 3.9a), both estimators are decent estimators for p_{ijt}, but the estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator. The uniform estimator (MSE: 0.0691) performs slightly better than the ratio estimator (MSE: 0.0717).

If the polling population and the voting population are significantly different and the riding probability deviates significantly from the regional probability (see figure 3.9b), the estimated p_{ijt} for both estimators are unreliable. The estimated standard error for the uniform estimator is more reliable than that for the ratio estimator. The uniform estimator (MSE: 2.5273) performs better than the ratio estimator (MSE: 2.7660).

Estimating number of votes

If the polling population and the voting population are slightly different and the riding probability deviates slightly from the regional probability (see figure 3.9a), both estimators are decent estimators for v_{ijt}, but the estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator. The uniform estimator (MSE: 6937826) performs slightly better than the ratio estimator (MSE: 7255206).

If the polling population and the voting population are significantly different and the riding probability deviates significantly from the regional probability (see figure 3.9b), the estimated v_{ijt} for both estimators are unreliable. The estimated standard error for the uniform estimator is more reliable than that for the ratio estimator. The uniform estimator (MSE: 253990617) performs better than the ratio estimator (MSE: 288694086).

Setting 7: Assumptions 1 and 2 violated, Assumption 3 holds

Estimating probability of votes

If the riding probability and the regional probability are slightly different and the number of voters between the two elections are slightly different (see figure 3.10a), both estimators provide reliable estimates for p_{ijt}. The estimated standard error for the uniform estimator are reliable except in region 3. The estimated standard error for the ratio estimator are not reliable. The uniform estimator (MSE: 0.0473) performs slightly better than the ratio estimator (MSE: 0.0477).

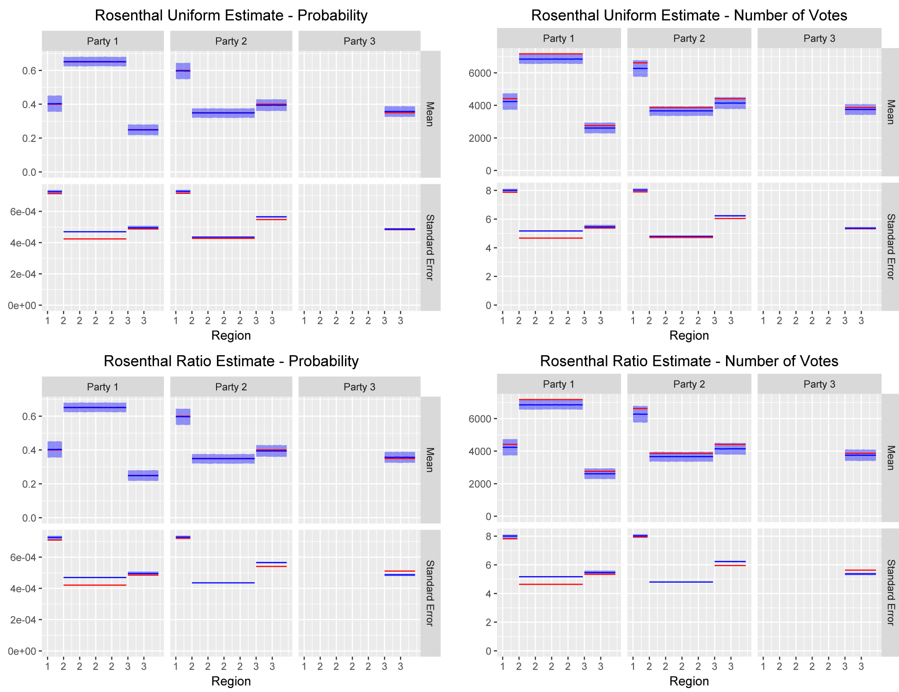

Figure 3.10a: Simulation results of Setting 7 ($\gamma = 0, \delta = 0.5, \phi = 0.05$)

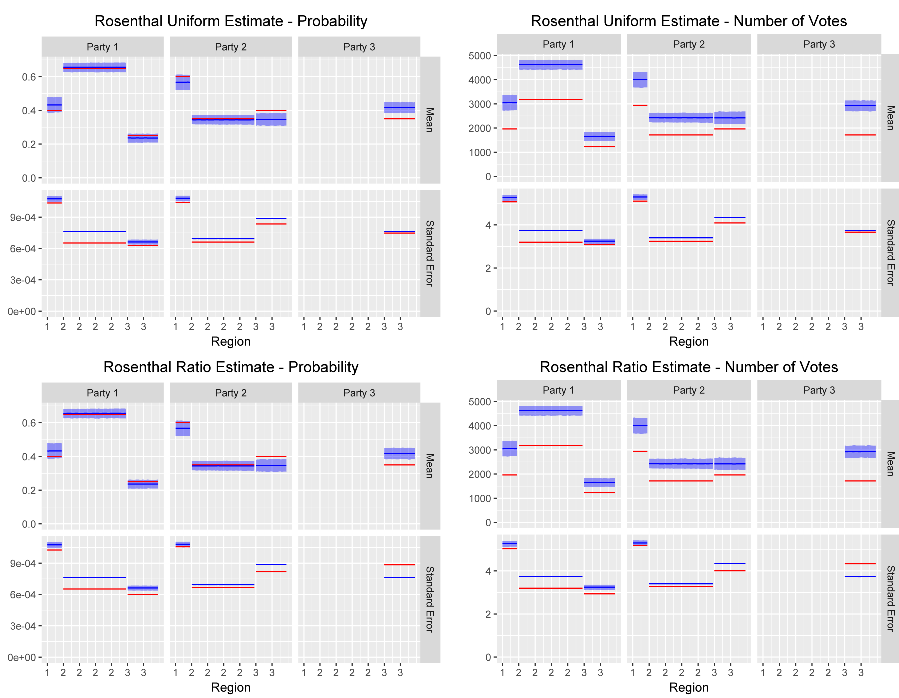

Figure 3.10b: Simulation results of Setting 7 ($\gamma = 0, \delta = 5, \phi = -0.3$)

If the riding probability and the regional probability are significantly different and the number of voters between the two elections are significant different (see Figure 3.10b), both estimators provide decent estimates for p_{ijt}. The estimated standard error for both estimators are not good. The uniform estimator (MSE: 0.2236) performs slightly better than the ratio estimator (MSE: 0.2242).

Estimating probability of votes

If the riding probability and the regional probability are slightly different and the number of voters between the two elections are slightly different (see figure 3.10a), the true v_{ijt} are contained in the 2.5 to 97.5 percentiles. The estimated standard error for the uniform estimator are reliable except in region 3. The estimated standard error for the ratio estimator are not reliable. The uniform estimator (MSE: 14251188) performs slightly better than the ratio estimator (MSE: 14309813).

If the riding probability and the regional probability are significantly different and the number of voters between the two elections are significant different (see Figure 3.10b), neither estimators are good for v_{ijt}. The estimated standard error for both estimators are not good. The uniform estimator (MSE: 165687785) performs slightly better than the ratio estimator (MSE: 165745644).

Setting 8: All of Assumptions 1, 2, and 3 violated

Estimating probability of votes

If all the assumptions were slightly violated (see figure 3.11a), both estimators are good estimators for p_{ijt}. The estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator but it is still not good. The uniform estimator (MSE: 0.0719) performs better than the ratio estimator (MSE: 0.0747).

If all the assumptions were significantly violated (see figure 3.11b), neither estimators are good estimators for p_{ijt}, but the estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator. The uniform estimator (MSE: 2.5208) performs better than the ratio estimator (MSE: 2.7609).

Estimating number of votes

If all the assumptions were slightly violated (see figure 3.11a), neither estimator is good for v_{ijt}. The estimated standard error for the uniform estimator is better than the estimated

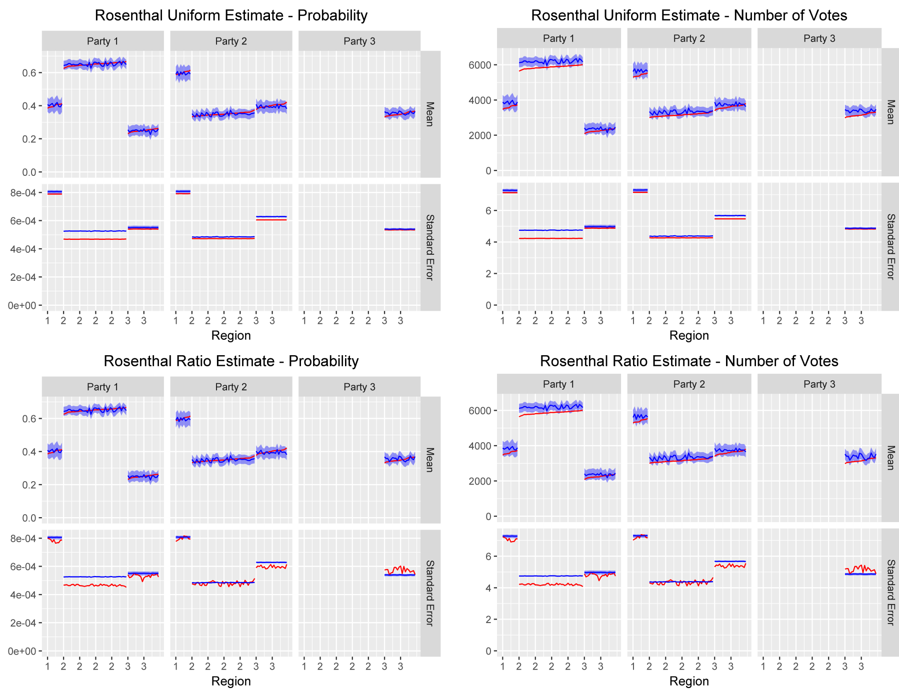

Figure 3.11a: Simulation results of Setting 8 ($\gamma = -1, \delta = 0.5, \phi = 0.05$)

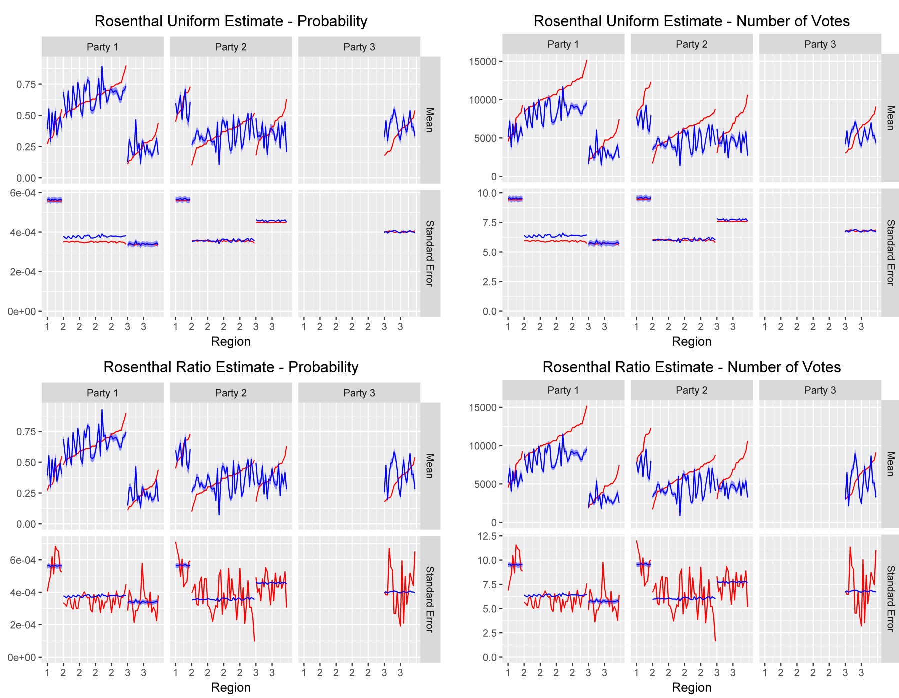

Figure 3.11b: Simulation results of Setting 8 ($\gamma = 10, \delta = 5, \phi = 0.3$)

standard error for the ratio estimator but it is still not good. The uniform estimator (MSE: 14227502) performs better than the ratio estimator (MSE: 14397981).

If all the assumptions were significantly violated (see figure 3.11b), neither estimators are good estimators for v_{ijt}, but the estimated standard error for the uniform estimator is better than the estimated standard error for the ratio estimator. The uniform estimator (MSE: 1081022150) performs better than the ratio estimator (MSE: 1140945250).

3.5 Summary

In each figure, there are four sets of plots. The plots to the left show the mean and histogram of the percent of correct calls for each iteration. If the estimator is a good estimator, the mean should be near 100% and the majority of the iterations should be able to achieve 100% correct calls.

The plots to the right show the mean and histogram of the estimated number of seats each party would obtain for each iteration. If the estimator is a good estimator, the mean should be close to the real number of seats.

Setting 1: All of Assumptions 1, 2, and 3 hold

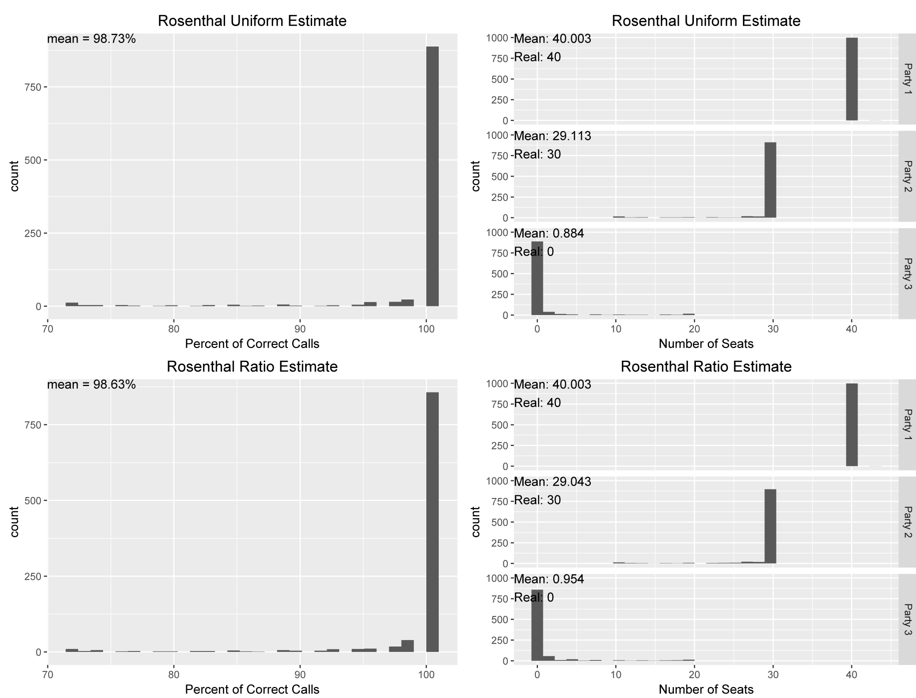

Figure 3.12: Simulation results of Setting 1 ($\gamma = 0, \delta = 0, \phi = 0$)

In the ideal case, both estimators are able to make the correct call around 99% of the time. The average number of estimated seats is within 1 seat of the real outcome (see figure 3.12).

Setting 2: Assumption 1 violated, Assumptions 2 and 3 hold

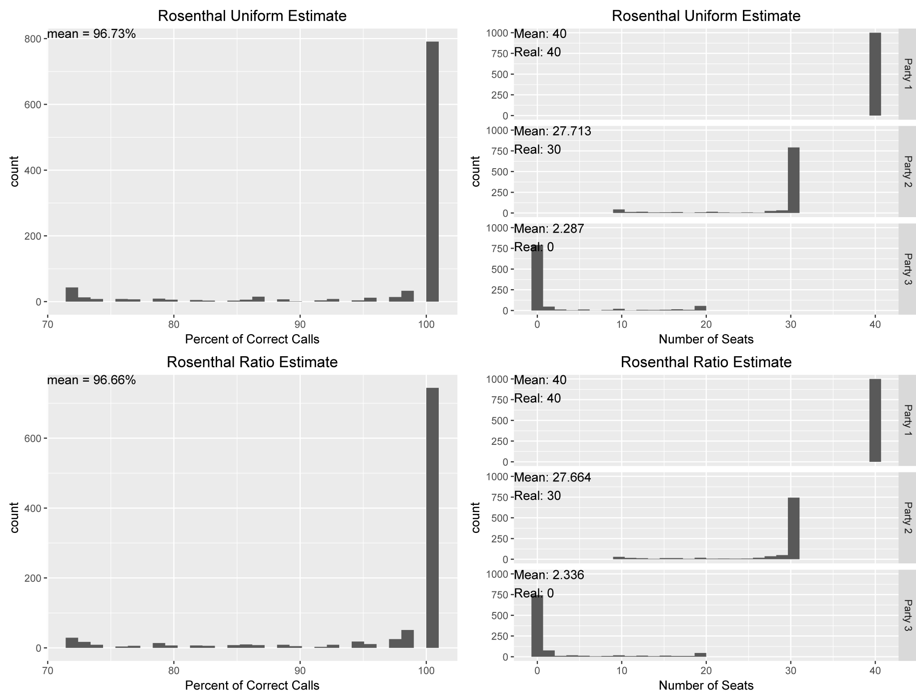

Figure 3.13a: Simulation results of Setting 2 ($\gamma = 0, \delta = 0.5, \phi = 0$)

If there is a slight difference in the polling population and the voting population, the estimators are able to make the correct call around 97% of the time, which is about 2% lower than in the ideal case. Also, the average number of estimated seats is within 3 seats of the real outcome (see figure 3.13a).

If there is a significant difference in the polling population and the voting population, the estimators make the correct calls around 72% of the time, which is 27% lower than in the ideal case. The average number of estimated seats is within 20 seats of the real outcome (see figure 3.13b).

Setting 3: Assumption 2 violated, Assumptions 1 and 3 hold

If there is a small difference in the number of voters between the two elections, the estimators make the correct calls around 99% of the time, which is about the same as the ideal case.

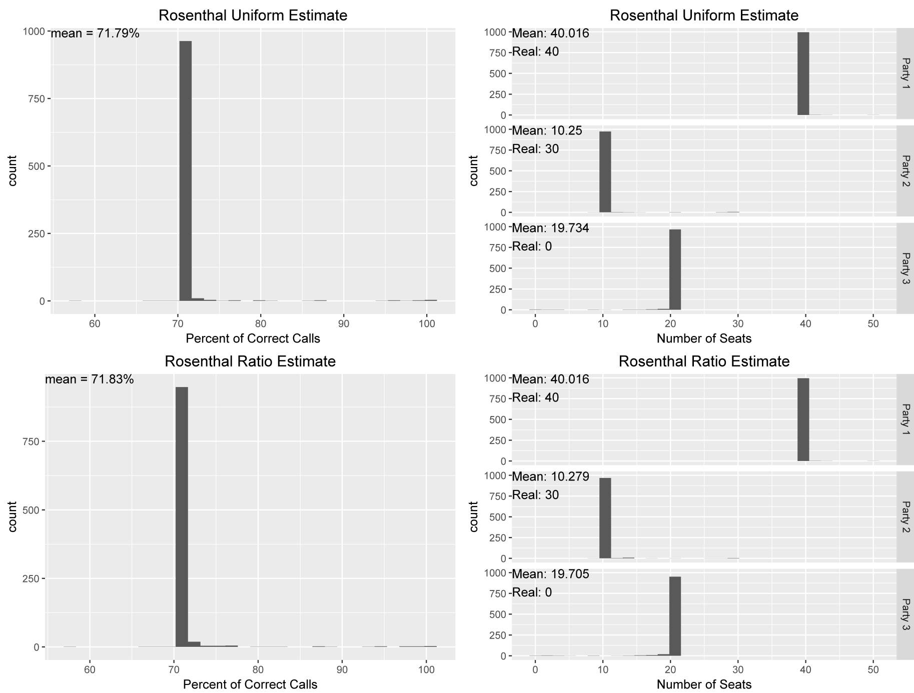

Figure 3.13b: Simulation results of Setting 2 ($\gamma = 0, \delta = 5, \phi = 0$)

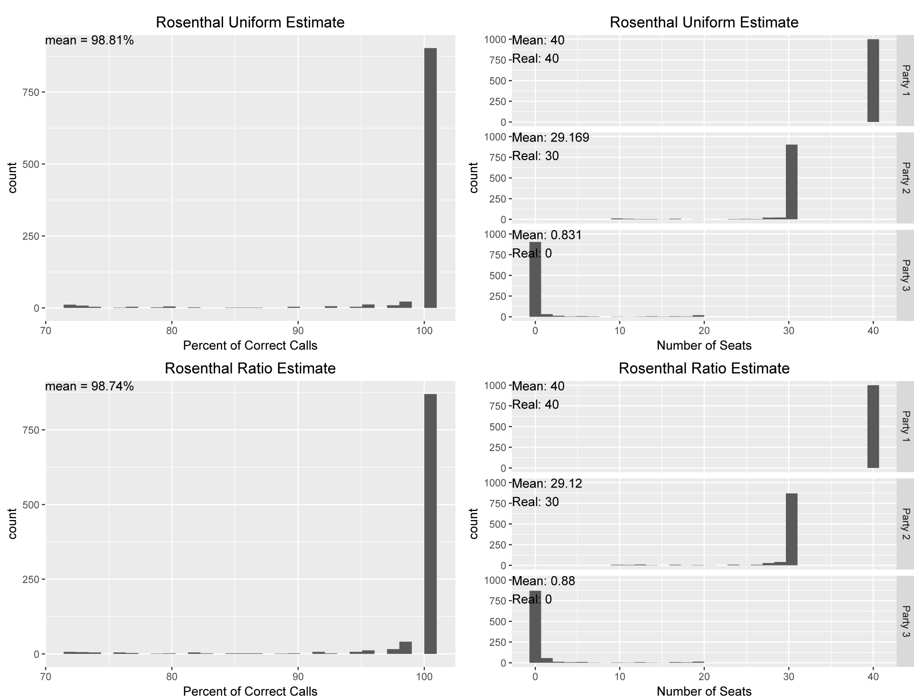

Figure 3.14a: Simulation results of Setting 3 ($\gamma = 0, \delta = 0, \phi = 0.05$)

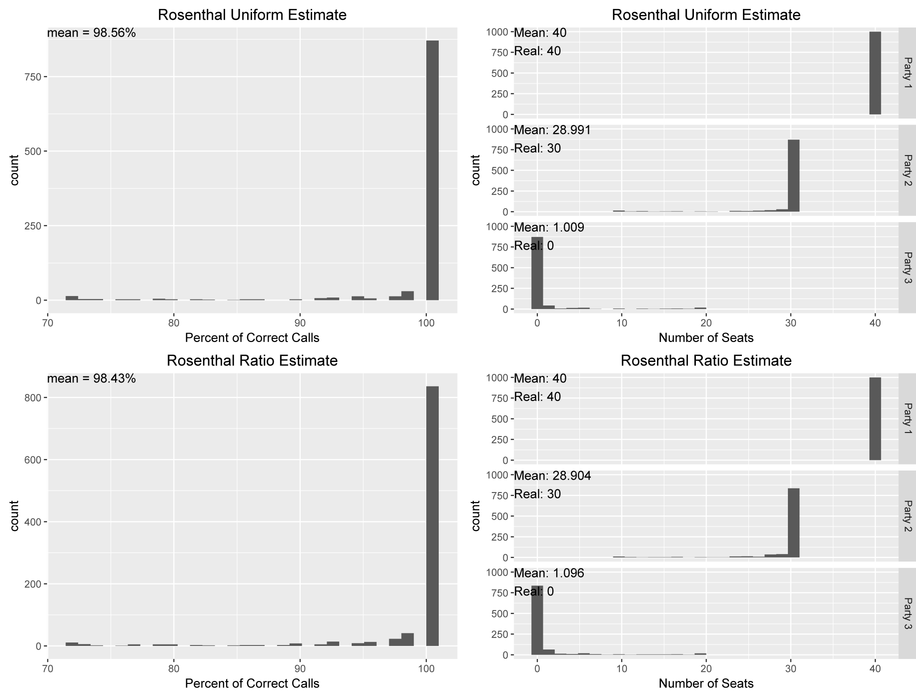

Figure 3.14b: Simulation results of Setting 3 ($\gamma = 0, \delta = 0, \phi = -0.3$)

The average number of estimated seats is within 1 seat of the real outcome (see figure 3.14a).

If there is a large difference in the number of voters between the two elections, the estimators make the correct calls around 98% of the time, which is about 1% less than the ideal case. The average number of estimated seats is around 1 seat of the real outcome (see figure 3.14b).

Setting 4: Assumption 3 violated, Assumptions 1 and 2 hold

If there is a slight difference in the riding probability and the regional probability, the estimators make the correct calls about 98% of the time, which is 1% less than the ideal case. Also, the average number of estimated seats is within 2 seats of the real outcome (see figure 3.15a).

If there is a significant difference in the riding probability and the regional probability, the estimators make the correct calls about 75% of the time, which is 24% less than the ideal case. The average number of estimated seats is within 1 seat of the real outcome (see figure 3.15b).

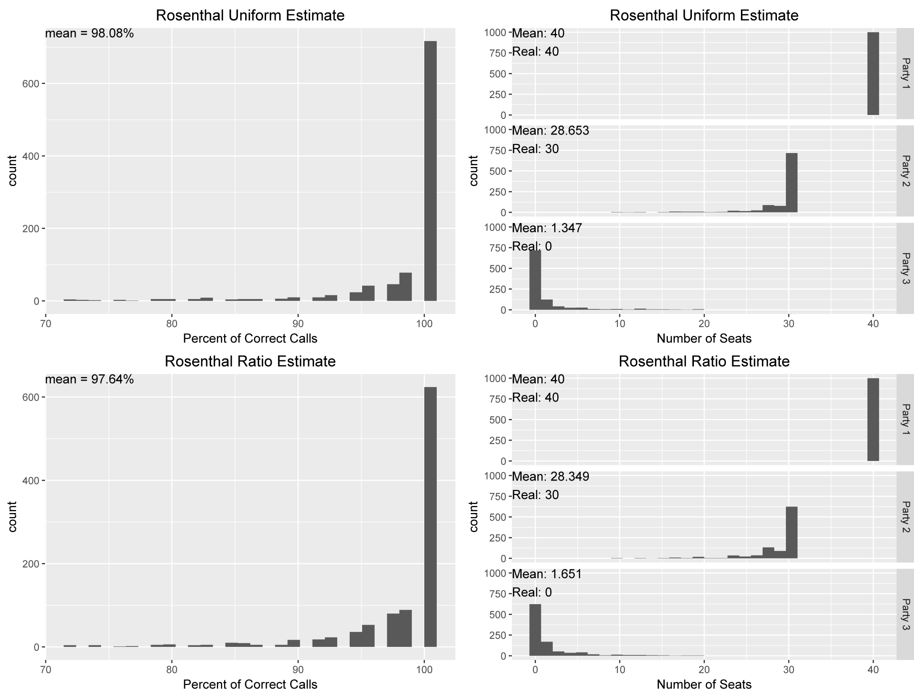

Figure 3.15a: Simulation results of Setting 4 ($\gamma = 1, \delta = 0, \phi = 0$)

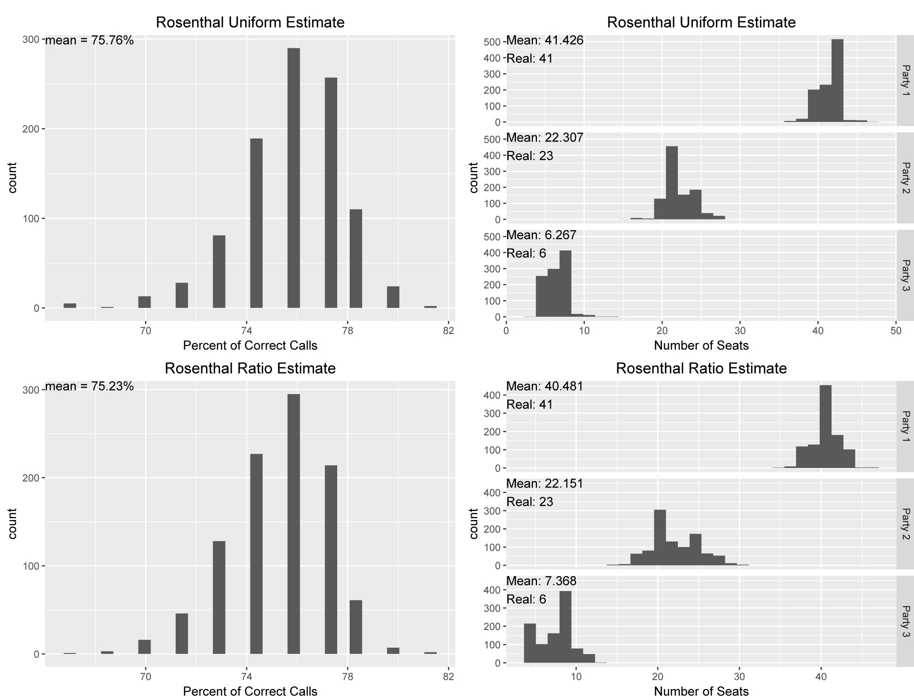

Figure 3.15b: Simulation results of Setting 4 ($\gamma = -10, \delta = 0, \phi = 0$)

Setting 5: Assumptions 2 and 3 violated, Assumption 1 holds

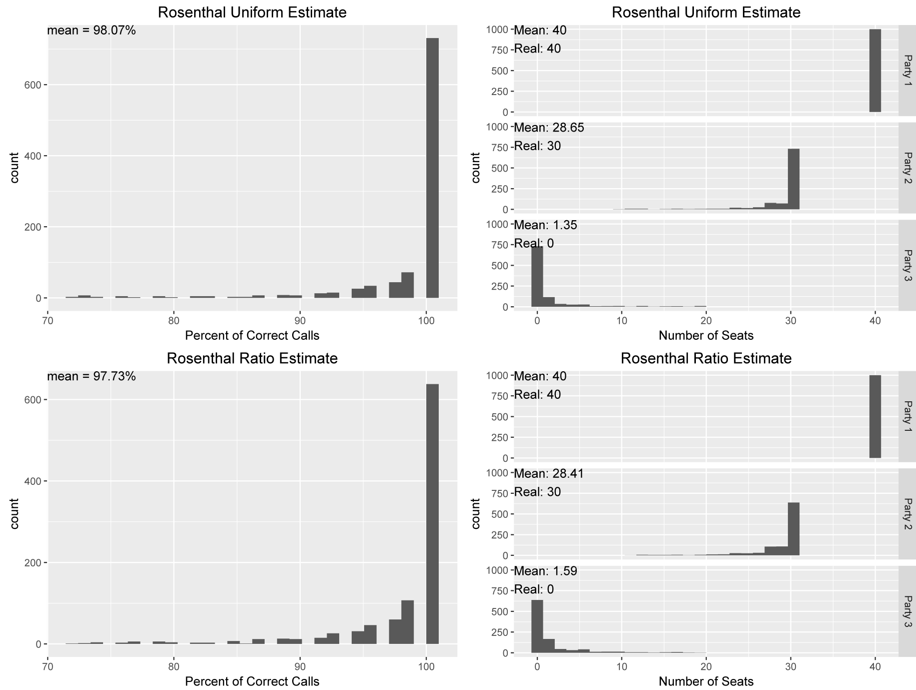

Figure 3.16a: Simulation results of Setting 5 ($\gamma = 1, \delta = 0, \phi = 0.05$)

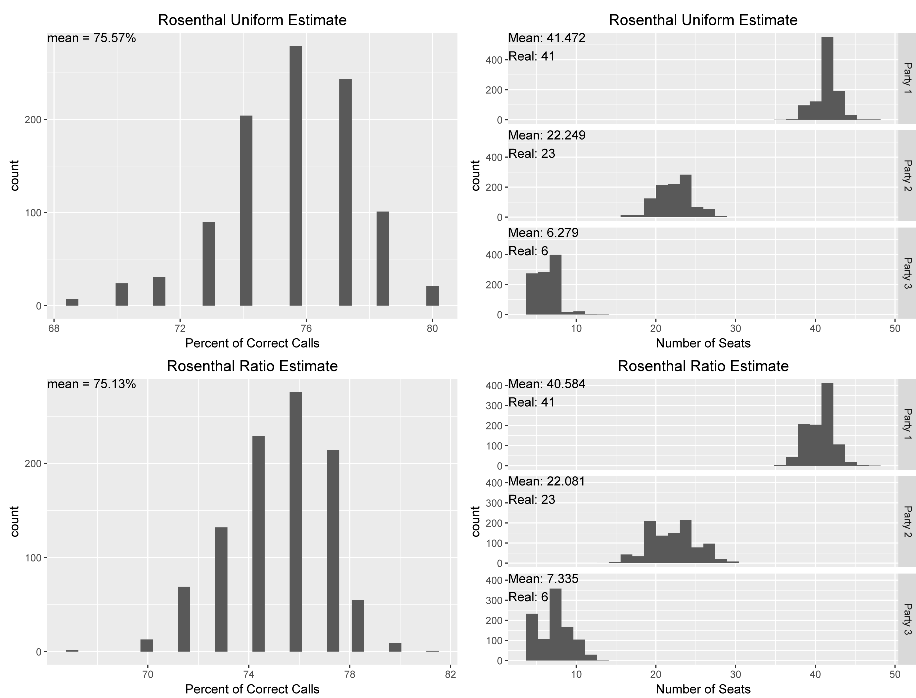

Figure 3.16b: Simulation results of Setting 5 ($\gamma = -10, \delta = 0, \phi = -0.3$)

If the number of voters between the two elections are slightly different and the riding probability is slightly different than the regional probability, the estimators make the correct calls about 98% of the time, which is 1% lower than the ideal case. Also, the average number of estimated seats is within 2 seats of the real outcome (see figure 3.16a).

If the number of voters between the two elections are significantly different and the riding probability is significantly different than the regional probability, the estimators make the correct calls about 75% of the time, which is 24% lower than the ideal case. Also, the average number of estimated seats is within 1 seat of the real value (see figure 3.16b).

Setting 6: Assumptions 1 and 3 violated, Assumption 2 holds

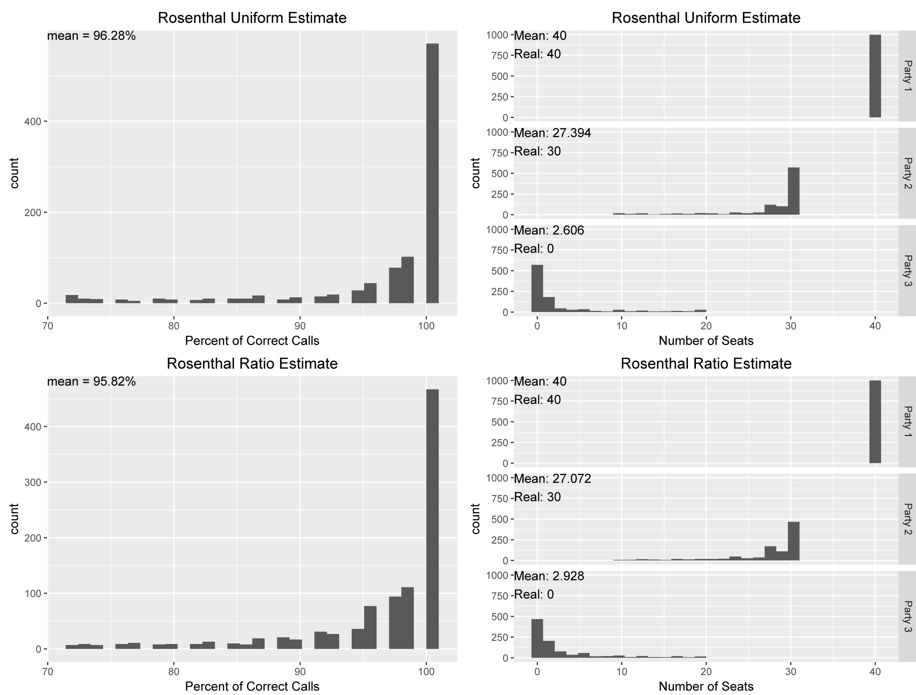

Figure 3.17a: Simulation results of Setting 6 ($\gamma = 1, \delta = 0.5, \phi = 0$)

If the polling population and the voting population are slightly different and the riding probability deviates slightly from the regional probability, the estimators make the correct call about 96% of the time, which is 3% lower than the ideal case. Also, the average number of estimated seats is within 3 seats of the real outcome (see figure 3.17a).

If the polling population and the voting population are significantly different and the riding probability deviates significantly from the regional probability, the uniform estimator makes the correct calls about 75% of the time, which is 24% lower than the ideal case. The ratio estimator makes the correct calls around 72% of the time, which is 27% lower than the ideal

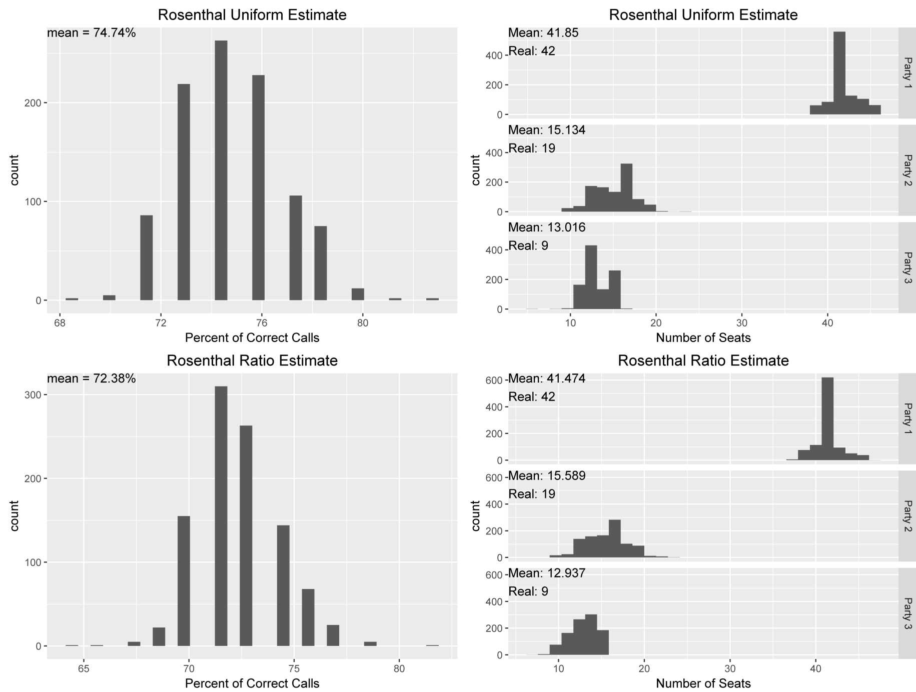

Figure 3.17b: Simulation results of Setting 6 ($\gamma = 10, \delta = 5, \phi = 0$)

case. The average number of estimated seats is within 4 seats of the real outcome for both estimators (see figure 3.17b).

Setting 7: Assumptions 1 and 2 violated, Assumption 3 holds

If the riding probability and the regional probability are slightly different and the number of voters between the two elections are slightly different, the estimators make the correct calls about 97% of the time, which is 2% lower than the ideal case. Also, the average number of estimated seats is within 2 seats of the real outcome (see figure 3.18a).

If the riding probability and the regional probability are significantly different and the number of voters between the two elections are significantly different, both estimators make the correct calls about 72% of the time, which is 27% lower than the ideal case. Also, the average number of estimated seats is within 20 seats of the real outcome (see figure 3.18b).

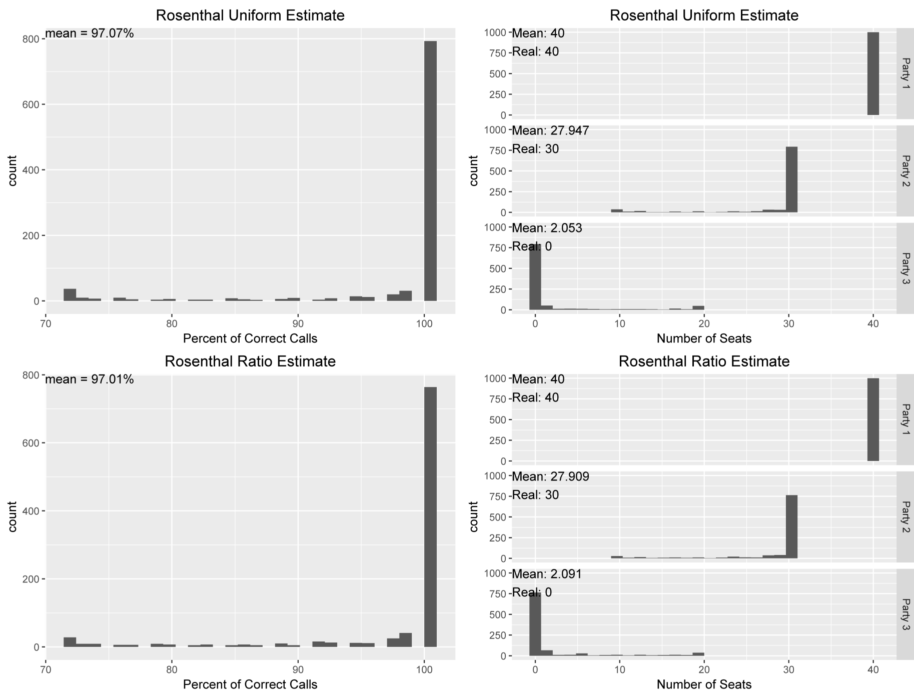

Figure 3.18a: Simulation results of Setting 7 ($\gamma = 0, \delta = 0.5, \phi = 0.05$)

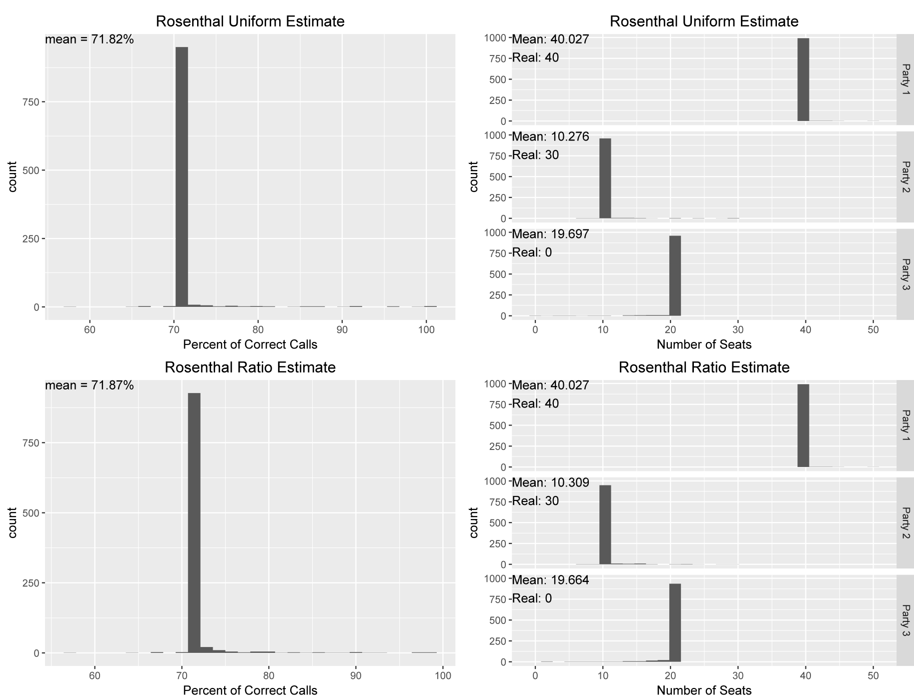

Figure 3.18b: Simulation results of Setting 7 ($\gamma = 0, \delta = 5, \phi = -0.3$)

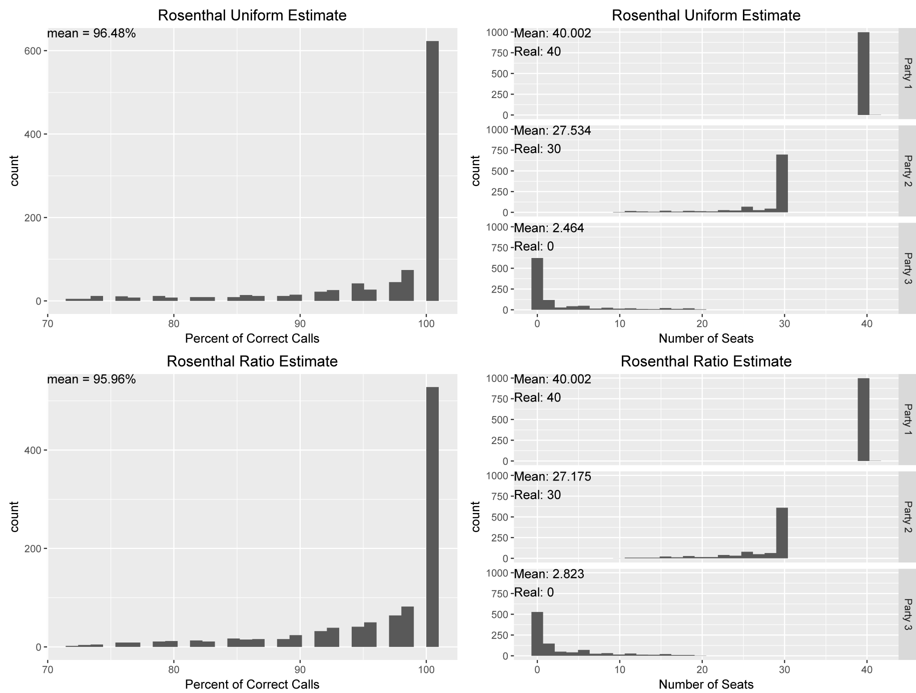

Figure 3.19a: Simulation results of Setting 8 ($\gamma = -1, \delta = 0.5, \phi = 0.05$)

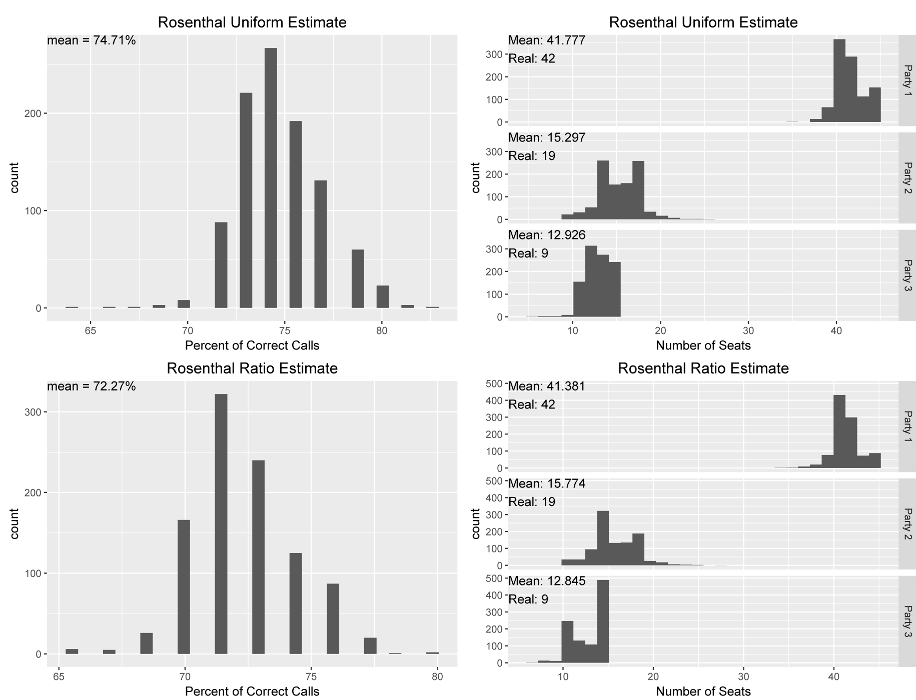

Figure 3.19b: Simulation results of Setting 8 ($\gamma = 10, \delta = 5, \phi = 0.3$)

Setting 8: All of Assumptions 1, 2, and 3 violated

If all the assumptions were slightly violated, both estimators make the correct calls about 96% of the time, which is 3% lower than the ideal case. The average number of estimated seats is within 3 seats of the real outcome (see figure 3.19a).

If all the assumptions were significantly violated, the uniform estimator makes the correct calls around 75% of the time, which is 24% lower than the ideal case. The ratio estimator makes the correct calls around 72% of the time, which is 27% lower than the ideal case. The average number of estimated seats is within 4 seats of the real value for both estimators (see figure 3.19b).

Summary

Table 3.5: Summary of all 8 simulation settings.

		Simulation Setting							
		1	2	3	4	5	6	7	8
p_{ijt}		✓		✓					
v_{ijt}		✓							
Percentage of Ridings Where the Estimators Correctly Predicts the Winner		99	72	98	75	75	75 (U) 72 (R)	72	75 (U) 72 (R)
Real Number of Seats	Party 1	40	40	40	41	41	42	40	42
	Party 2	30	30	30	23	23	19	30	19
	Party 3	0	0	0	6	6	9	0	9
Average Estimated Number of Seats	Party 1	40	40	40	41 (U) 40 (R)	41	42 (U) 41 (R)	40	42 (U) 41 (R)
	Party 2	29	10	29	22	22	15 (U) 16 (R)	10	15 (U) 16 (R)
	Party 3	1	20	1	6 (U) 7 (R)	7	13	20	13
Difference Between Estimated and Real Number of Seats	Party 1	0	0	0	0 (U) 1 (R)	0	0 (U) 1 (R)	0	0 (U) 1 (R)
	Party 2	1	20	1	1	1	4 (U) 3 (R)	20	4 (U) 3 (R)
	Party 3	1	20	1	0 (U) 1 (R)	1	4	20	4

Table 3.5 shows the summary of all 8 simulation settings. p_{ijt} indicates whether the Rosenthal estimators are able to predict the probability of a voter voting for party j in riding i well. v_{ijt} indicates whether the estimators can predict the number of votes well. The percentage of ridings where the estimator correctly predicts the winner is shown in the

third row. The real and average estimated number of seats are shown as well. The average estimated number of seats is obtained by calculating the average of the simulated results. The absolute difference between the real and average estimated number of seats is shown in the last three rows.

If the the voting pattern of the people who participate in polls and the voters who vote on election day are significantly different than the previous election (simulation setting 2), the probability and number of votes cannot be well estimated, the percent of correct calls drops to 72%, and the estimated number of seats is very different than the real election outcome. Rosenthal's estimators rely heavily on the validity of this assumption.

If the number of voters is drastically different than the previous election (simulation setting 3), the number of votes cannot be well estimated, but the estimated probability of votes is still reliable, the percent of correct calls is almost as high as the ideal case, and the estimated number of seats is very accurate. Rosenthal's estimators are least dependent on this assumption.

If the riding probabilities are significantly different than the regional probability (simulation setting 4), neither the probability nor the number of votes can be well estimated, the percent of correct call drops to 75%, but the estimated number of seats is still very accurate. Rosenthal's estimators can still provide reasonable estimate at a national level (number of seats), but the results could be very misleading at a riding level, as seen in the low percent of ridings in which the estimator correctly predicts the winner.

Table 3.6: Comparison for selected cases in all 8 simulation settings. Please see Appendix E for the complete list of simulation outcomes in all settings.

				$MSE_{Uniform}/MSE_{Ratio}$		
Setting	γ	δ	ϕ	p_{ijt}	v_{ijt}	Recommend
1	0	0	0	0.988	0.988	Uniform
2	0	0.5	0	0.992	0.988	Uniform
2	0	5	0	0.998	0.996	Uniform
3	0	0	0.05	0.991	0.996	Uniform
3	0	0	-0.3	0.985	1.000	Uniform
4	1	0	0	0.964	0.956	Uniform
4	-10	0	0	0.910	0.958	Uniform
5	1	0	0.05	0.964	0.981	Uniform
5	-10	0	-0.3	0.910	0.971	Uniform
6	1	0.5	0	0.964	0.956	Uniform
6	10	5	0	0.914	0.880	Uniform
7	0	0.5	0.05	0.992	0.996	Uniform
7	0	5	-0.3	0.997	1.000	Uniform
8	-1	0.5	-0.05	0.963	0.988	Uniform
8	10	5	0.3	0.913	0.947	Uniform

Based on table 3.6, The performance of Rosenthal's two estimators are very similar. They tend to do well together, or they will perform poorly together.

3.6 Significance of the Simulation Results

3.6.1 Performance of Estimators Under Violated Assumptions

Estimating the probability of votes

If there is a large difference between the voting pattern of the polls and the voters, the estimated p_{ijt} will not be reliable, but the estimated standard deviation will still be reliable to a certain extent. If there is a large difference between the riding probability and the regional probability, neither estimator will be able to estimate the p_{ijt} well, but the estimated variances are still valid for the uniform estimator. The estimated p_{ijt} and the estimated standard deviation of p_{ijt} for both estimators do not seem to be affected by a change in the number of voters.

If it is known that there is a large difference between the riding probabilities and the regional probability, a large difference between the voting pattern of the polls and voters does not seem to make the estimated p_{ijt} any worse. If it is known that the number of voters are different between the elections, a large difference in the voting pattern of the polls and voters makes the estimated p_{ijt} unreliable. A large difference in the number of voters between the elections does not seem to make the estimated p_{ijt} worse if it is known that any one of the other assumptions are violated. If it is known that the number of voters are different between the elections, p_{ijt} will not be well estimated if there is also a large difference between the riding probabilities and the regional probability. If it is known that there is a large difference in the voting pattern of the polls and voters, a large difference between the riding probabilities and the regional probability will not make the estimated p_{ijt} any worse. If it is known that any two assumptions are violated, the violation of the third assumption will not make the estimated p_{ijt} any worse.

Estimating the number of votes

If there is a large difference between the polls and the voters, the estimated v_{ijt} will not be reliable, but the estimated standard deviation will still be reliable to a certain extent. If there is a large difference between the riding probability and the regional probability, neither estimator will be able to estimate the v_{ijt} well, but the estimated variances are still valid for the uniform estimator. The estimated v_{ijt} is really sensitive to a change in the number of voters. Even slight deviation will cause the 2.5 to 97.5 percentile of the estimated v_{ijt} to not cover the true value.

If it is known that any one of the three assumptions are violated, the estimated v_{ijt} does not seem to be affected if one or two more assumptions are also violated.

Percentage of Correct Calls

If there is a large difference between the voting pattern of the polls and the voters, both the uniform and ratio estimators make the correct calls about 72% of the time. If there is a large difference between the riding probability and the regional probability, both estimators make the correct calls about 75% of the time. If there is a large difference in the number of voters between the two elections, both estimators make the correct calls about 98% of the time.

If it is known that there is a large difference between the riding probabilities and the regional probability, a large difference between the voting pattern of the pollers and voters decreases the percent of correct calls by 1% for the uniform estimator and 3% for the ratio estimator. If it is known that the number of voters are different between the elections, a large difference in the voting pattern of the pollers and voters does not change the percent of correct calls. A large difference in the number of voters between the elections does not seem to have an effect on the percent of correct calls if it is known that any one of the other assumptions are violated. If it is known that the number of voters are different between the elections, the percent of correct calls will decrease by 27% for both estimators if there is also a large difference between the riding probabilities and the regional probability. If it is known that there is a large difference in the voting pattern of the polls and voters, a large difference between the riding probabilities and the regional probability increases the percent of correct calls by 3% for the uniform estimator and has no effect on the percent of correct calls for the ratio estimator.

A large difference in the riding probabilities and the regional probability will increase the percent of correct calls for the uniform estimator by 3% and has no effect on the ratio estimator if it is known that the other two assumptions are violated. A difference in the voting patterns between the polls and voters decreases the percent of correct calls by 1% for the uniform estimator and 3% for the ratio estimator if it is known that the other two assumptions are also violated. A large difference in the number of voters between the two elections has no effect on the percent of correct calls for both estimators if it is known that the other two assumptions are also violated.

Estimating the Number of Seats

If there is a large difference between the polls and the voters, the average estimated number of seats is within 20 of the real value. If there is a large difference between the riding probability and the regional probability, the average estimated number of seats is within 1 seat from the real value for the uniform estimator and within 2 seats for the ratio estimator. Even though the average estimated number of seats is very close to the real value, the estimated number of seats spreads over a large range of number. If there is a large difference in the number of voters between the two elections, the average estimated number of seats is within 1 seat for both the uniform and ratio estimator.

If it is known that there is a large difference between the riding probabilities and the regional probability, a large difference between the voting pattern of the polls and voters will result in the average estimated number of seats to be within 4 seats for both estimators, instead of within 1 seat for the uniform estimator and within 2 seats for the ratio estimator. If it is known that the number of voters are different between the elections, a large difference in the voting pattern of the polls and voters will result in the average estimated number of seats to be within 20 seats for both estimators, instead of within 1 seat for both estimators. A large difference in the number of voters between the elections does not seem to have an effect on the average estimated number of seats if it is known that any one of the other assumptions are violated. If it is known that the number of voters are different between the elections, if there is also a large difference between the riding probabilities and the regional probability, the average estimated number of seats will be within 1 seat for the uniform estimator and 2 seats for the ratio estimator, instead of 1 seat for both estimators. If it is known that there is a large difference in the voting pattern of the polls and voters, a large difference between the riding probabilities and the regional probability will result in the average estimated number of seats to be within 4 seats for both estimator, instead of within 20 seats.

A large difference in the riding probabilities and the regional probability will result in the average estimated number of seats to be within 4 seats for both estimators instead of within 20 seats for both estimators if it is known that the other two assumptions are violated. A difference in the voting patterns between the polls and voters will result in the average estimated number of seats to be within 4 seats of both estimators instead of within 1 seat for the uniform estimator and 2 seats for the ratio estimator if it is known that the other two assumptions are also violated. A large difference in the number of voters between the two elections has no effect on the average estimated number of seats if it is known that the other two assumptions are also violated.

Summary

The predicted outcome of an election using the estimators proposed by Rosenthal is most adversely affected by the difference in voting pattern between the people who participate in polls and the people who votes on election day. The violation of this assumption will affect the estimated probability of votes, the estimated number of votes, the percentage of correct calls, and the estimated number of seats.

The predicted outcome of an election using Rosenthal's estimators is least adversely affected by the difference in the number of voters between the two consecutive elections. The violation of this assumption will affect the estimated number of votes, but it will only slightly affect the estimated probability of votes. The percentage of correct calls is around 98%, which is nearly all correct. The estimated number of seats is within around 1 seat of the real value.

3.6.2 Application to Real Elections

In a real election, all three assumptions will be violated to various degrees. The uniform estimator performs better than the ratio estimator under all situations. If there is a significant difference in the results from the two estimators, the results from the uniform estimator should be trusted rather than the results from the ratio estimator.

Chapter 4

Discussion

4.1 Conclusion

In conclusion, Rosenthal's estimators work very well under specific situations. The estimators depend on three key assumptions. First, the difference in the population of people who participated in the polls and the population of people who will actually vote on election day is constant across Canada and is the same as the previous election. Second, the number of voters are the same as the previous election. Third, the probability of an individual voting for a certain party in a certain riding is the same as the probability of an individual voting for the party in the region.

As the results from the simulation study show, the violation of each of these assumptions affect the percentage of correct calls and the estimated seat counts to various degrees. These two measures are of special importance because these two measures are what the voters, candidates, and party truly want to predict in an election. If there is a difference in population of people who participate in the polls and the people who votes on election day, the percent of correct calls will be affected greatly, and the estimated number of seats will not be reliable. If there is a difference between the number of voters in the two elections, the percent of correct calls will not be affected much, and the estimated number of seats will be reliable. If there is a large difference between the riding probability and the regional probability, the percent of correct calls will be affected greatly, but the estimated number of seats will still be reliable.

Grenier included individual characteristics of each candidates and used information from more than one election in his estimation. Using the election data from 2006 to 2015, preliminary analysis shows that male candidates have a significantly higher probability of being elected (OR: 1.29 (1.12, 1.49)). It also shows that incumbents have a significantly higher probability of being elected (OR: 30.68 (26.01, 36.20)). These are important predictors that would affect the outcome of an election. Unfortunately, the use of candidate level information and more election data was not examined in this project.

4.2 Future Research

The variance estimators presented in this study do not perform well, even under the scenario where all three assumptions are satisfied. The future research includes improving the variance estimators of Rosenthal's estimators and using them to provide predictive intervals.

Grenier's approach involves combining information from the previous three elections; however, no supporting literature can be found to justify the weights that he chose to use in his estimator. Future research includes a more thorough examination of Grenier's estimator via simulation, and to study how to improve its means of combining the previous three elections' information.

Lastly, after a more thorough understanding of Grenier's estimator, future research includes developing an alternative estimator by combining the advantages of Grenier's and Rosenthal's estimators. More specifically, this estimator should include overperforming effect, election data from multiple elections, and candidate level information.

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Appendix A

Derivation of the Variances

$$\mathbf{X}_{ijt} = \begin{bmatrix} M^*_{j\cdot 1,t-1} \\ M^*_{j\cdot 2,t-1} \\ M^*_{j\cdot 3,t-1} \\ M^*_{j\cdot l(i)t} \\ v_{1j,t-1} \\ \vdots \\ v_{Ij,t-1} \end{bmatrix}$$

$$f(\hat{\mathbf{X}}_{ijt}) \approx f(\mathbf{X}_{ijt}) + \left[\left(\hat{\mathbf{X}}_{ijt} - \mathbf{X}_{ijt}\right)^T \nabla f(\mathbf{X}_{ijt})\right]$$

$$\begin{aligned} Var(f(\hat{\mathbf{X}}_{ijt})) &\approx Var\left(f(\mathbf{X}_{ijt}) + \left[\left(\hat{\mathbf{X}}_{ijt} - \mathbf{X}_{ijt}\right)^T \nabla f(\mathbf{X}_{ijt})\right]\right) \\ &= Var\left(f(\mathbf{X}_{ijt}) + \sum_k \left(\hat{\mathbf{X}}_{ijt} - \mathbf{X}_{ijt}\right) \nabla f(\mathbf{X}_{ijt})_k\right) \\ &= \sum_a \left\{ \left[\frac{\partial f(\mathbf{X}_{ijt})}{\partial X_{ijta}}\right]^2 Var(X_{ijta}) + \sum_{m<a} 2Cov(X_{ijtm}, X_{ijta}) \right\} \end{aligned}$$

$$
\begin{aligned}
\hat{v}_{ijt,U} &= v_{ij,t-1} + \Big(\hat{p}^*_{jl(i)t} - r^*_{jl(i),t-1} + (r_{j,t-1} - \hat{p}_{j,t-1})\Big) \cdot n_{i,t-1} \\
&= v_{ij,t-1} + \left(\frac{\sum_k M^*_{jkl(i)t}}{\sum_k s^*_{kl(i)t}} - \frac{v^*_{jl(i),t-1}}{n^*_{l(i),t-1}} + \left(\frac{\sum_i v_{ij,t-1}}{\sum_i n_{i,t-1}} - \frac{\sum_l \sum_k M^*_{jkl,t-1}}{\sum_l \sum_k s^*_{kl,t-1}}\right)\right) \cdot n_{i,t-1} \\
&= v_{ij,t-1} + \left(\frac{M^*_{j\cdot l(i)t}}{s^*_{\cdot l(i)t}} - \frac{\sum_{l(x)=l(i)} v_{xj,t-1}}{\sum_{l(x)=l(i)} n_{x,t-1}} + \left(\frac{\sum_i v_{ij,t-1}}{\sum_i n_{i,t-1}} - \frac{\sum_l M^*_{j\cdot l,t-1}}{\sum_l s^*_{\cdot l,t-1}}\right)\right) \cdot n_{i,t-1} \\
\hat{v}_{ijt,R} &= v_{ij,t-1} \frac{\hat{p}^*_{jl(i)t} + (r_{j,t-1} - \hat{p}_{j,t-1})}{r^*_{jl(i),t-1}} \\
&= v_{ij,t-1} \left(\frac{\frac{\sum_k M^*_{jkl(i)t}}{\sum_k s^*_{kl(i)t}} + \left(\frac{\sum_i v_{ij,t-1}}{\sum_i n_{i,t-1}} - \frac{\sum_l M^*_{j\cdot l,t-1}}{\sum_l s^*_{\cdot l,t-1}}\right)}{\frac{\sum_{l(x)=l(i)} v_{xj,t-1}}{\sum_{l(x)=l(i)} n_{x,t-1}}}\right) \\
&= \left(\frac{v_{ij,t-1} \sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1}}\right) \left[\frac{M^*_{j\cdot l(i)t}}{s^*_{\cdot l(i)t}} + \left(\frac{\sum_i v_{ij,t-1}}{\sum_i n_{i,t-1}} - \frac{\sum_l M^*_{j\cdot l,t-1}}{\sum_l s^*_{\cdot l,t-1}}\right)\right]
\end{aligned}
$$

Rosenthal Uniform Estimator

$$
\frac{\partial \hat{v}_{ijt,U}}{\partial X_{ijta}} = \begin{bmatrix} -\frac{n_{i,t-1}}{\sum_l \sum_k s^*_{kl,t-1}} \\ -\frac{n_{i,t-1}}{\sum_l \sum_k s^*_{kl,t-1}} \\ -\frac{n_{i,t-1}}{\sum_l \sum_k s^*_{kl,t-1}} \\ \frac{n_{i,t-1}}{\sum_k s^*_{kl(i)t}} \\ \frac{\partial \hat{v}_{ijt,U}}{\partial \hat{v}_{1j,t-1}} \\ \vdots \\ \frac{\partial \hat{v}_{ijt,U}}{\partial \hat{v}_{Ij,t-1}} \end{bmatrix}
\quad
Var(X_{ijta}) = \begin{cases} \sum_{k=1}^{9} s^*_{k1,t-1} q^*_{jk1,t-1}(1 - q^*_{jk1,t-1}) & \text{if } a = 1 \\ \sum_{k=1}^{9} s^*_{k2,t-1} q^*_{jk2,t-1}(1 - q^*_{jk2,t-1}) & \text{if } a = 2 \\ \sum_{k=1}^{9} s^*_{k3,t-1} q^*_{jk3,t-1}(1 - q^*_{jk3,t-1}) & \text{if } a = 3 \\ \sum_{k=1}^{9} s^*_{kl(i)t} q^*_{jkl(i)t}(1 - q^*_{jkl(i)t}) & \text{if } a = 4 \\ n_{a-4,t-1} p_{a-4,j,t-1}(1 - p_{a-4,j,t-1}) & \text{if } a \geq 5 \end{cases}
$$

Rosenthal Ratio Estimator

$$\frac{\partial \hat{v}_{ijt,R}}{\partial X_{ijta}} = \begin{bmatrix} -\frac{v_{ij,t-1}\sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1}\sum_l s^*_{\cdot l,t-1}} \\ -\frac{v_{ij,t-1}\sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1}\sum_l s^*_{\cdot l,t-1}} \\ -\frac{v_{ij,t-1}\sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1}\sum_l s^*_{\cdot l,t-1}} \\ \frac{v_{ij,t-1}\sum_{l(x)=l(i)} n_{x,t-1}}{s^*_{\cdot l(i)t}\sum_{l(x)=l(i)} v_{xj,t-1}} \\ \frac{\partial \hat{v}_{ijt,R}}{\partial \hat{v}_{1j,t-1}} \\ \vdots \\ \frac{\partial \hat{v}_{ijt,R}}{\partial \hat{v}_{Ij,t-1}} \end{bmatrix} \quad Var(X_{ijta}) = \begin{cases} \sum_{k=1}^9 s^*_{k1,t-1} q^*_{jk1,t-1}(1-q^*_{jk1,t-1}) & \text{if } a=1 \\ \sum_{k=1}^9 s^*_{k2,t-1} q^*_{jk2,t-1}(1-q^*_{jk2,t-1}) & \text{if } a=2 \\ \sum_{k=1}^9 s^*_{k3,t-1} q^*_{jk3,t-1}(1-q^*_{jk3,t-1}) & \text{if } a=3 \\ \sum_{k=1}^9 s^*_{kl(i)t} q^*_{jkl(i)t}(1-q^*_{jkl(i)t}) & \text{if } a=4 \\ n_{a-4,t-1}p_{a-4,j,t-1}(1-p_{a-4,j,t-1}) & \text{if } a\geq 5 \end{cases}$$

$$G = \frac{g_1}{g_2} = \frac{v_{ij,t-1}\sum_{l(x)=l(i)} n_{x,t-1}}{\sum_{l(x)=l(i)} v_{xj,t-1}}$$
$$H = \frac{M^*_{j\cdot l(i)t}}{s^*_{\cdot l(i)}} + \frac{\sum_i v_{ij,t-1}}{\sum_i n_{i,t-1}} - \frac{\sum_l M^*_{j\cdot l,t-1}}{\sum_l s^*_{\cdot l,t-1}}$$

$$\frac{\partial \hat{v}_{ijt,R}}{\partial \hat{v}_{xj,t-1}} = \begin{cases} \left(\frac{(g_2\sum_{l(i)=l} n_{i,t-1})-g_1}{g_2^2}\right)\cdot H + \frac{G}{\sum_i n_{i,t-1}} & \text{if } x=i \\ \left(\frac{-g_1}{g_2^2}\right)\cdot H + \frac{G}{\sum_i n_{i,t-1}} & \text{if } l(x)=l(i) \text{ and } x\neq i \\ \frac{G}{\sum_i n_{i,t-1}} & \text{if } l(x)\neq l(i) \end{cases}$$

Appendix B

List of β_{ijt} Used in the Simulation

Riding	Party 1	Party 2	Party 3
1	-0.008	0.008	0
2	0.006	-0.006	0
3	-0.005	0.005	0
4	0.009	-0.009	0
5	-0.013	0.013	0
6	-0.009	0.009	0
7	0.005	-0.005	0
8	-0.008	0.008	0
9	0.008	-0.008	0
10	0.015	-0.015	0
11	0.011	-0.011	0
12	0.005	-0.005	0
13	-0.014	0.014	0
14	-0.006	0.006	0
15	-0.004	0.004	0
16	-0.008	0.008	0
17	-0.002	0.002	0
18	-0.017	0.017	0
19	-0.003	0.003	0
20	-0.002	0.002	0
21	0.009	-0.009	0
22	0.003	-0.003	0
23	-0.01	0.01	0
24	-0.004	0.004	0
25	0.025	-0.025	0
26	0.01	-0.01	0

Riding	Party 1	Party 2	Party 3
27	0.008	-0.008	0
28	-0.006	0.006	0
29	-0.005	0.005	0
30	-0.005	0.005	0
31	0.011	-0.011	0
32	0.004	-0.004	0
33	0.008	-0.008	0
34	-0.013	0.013	0
35	0.01	-0.01	0
36	-0.011	0.011	0
37	-0.01	0.01	0
38	0.005	-0.005	0
39	-0.004	0.004	0
40	0.001	-0.001	0
41	-0.011	0.011	0
42	0.016	-0.016	0
43	-0.002	0.002	0
44	0.002	-0.002	0
45	-0.012	0.012	0
46	0.002	-0.002	0
47	0.005	-0.005	0
48	0.02	-0.02	0
49	-0.008	0.008	0
50	0.002	-0.002	0
51	-0.007	-0.006	0.013
52	-0.011	0.006	0.005
53	0.003	-0.009	0.006
54	-0.005	0.009	-0.004
55	-0.002	-0.003	0.005
56	0.019	-0.01	-0.009
57	0.003	-0.022	0.019
58	0.005	-0.007	0.002
59	0.003	-0.015	0.012
60	-0.006	0.023	-0.017
61	0.013	0.001	-0.014
62	0.009	-0.007	-0.002
63	-0.011	0.008	0.003
64	0.004	0.012	-0.016
65	-0.002	-0.005	0.007
66	0.005	0.009	-0.014
67	0.007	0.006	-0.013
68	-0.009	0.001	0.008
69	-0.014	0.015	-0.001
70	-0.004	-0.006	0.01

Appendix C

List of $\beta_{ij,t-1}$ Used in the Simulation

Riding	Party 1	Party 2	Party 3
1	-0.014	0.014	0
2	-0.009	0.009	0
3	-0.005	0.005	0
4	0.006	-0.006	0
5	-0.004	0.004	0
6	0.012	-0.012	0
7	0.01	-0.01	0
8	0.009	-0.009	0
9	-0.002	0.002	0
10	-0.003	0.003	0
11	-0.003	0.003	0
12	-0.005	0.005	0
13	-0.008	0.008	0
14	0.003	-0.003	0
15	0.004	-0.004	0
16	-0.009	0.009	0
17	-0.012	0.012	0
18	0.003	-0.003	0
19	-0.011	0.011	0
20	-0.011	0.011	0
21	0.004	-0.004	0
22	0.005	-0.005	0
23	0.008	-0.008	0
24	0.013	-0.013	0
25	0.008	-0.008	0
26	0.004	-0.004	0

Riding	Party 1	Party 2	Party 3
27	0.005	-0.005	0
28	-0.013	0.013	0
29	-0.016	0.016	0
30	0.009	-0.009	0
31	-0.003	0.003	0
32	0.006	-0.006	0
33	0.003	-0.003	0
34	-0.017	0.017	0
35	0.001	-0.001	0
36	-0.011	0.011	0
37	-0.002	0.002	0
38	-0.002	0.002	0
39	0.012	-0.012	0
40	0.001	-0.001	0
41	-0.017	0.017	0
42	0.004	-0.004	0
43	-0.006	0.006	0
44	0.014	-0.014	0
45	0.004	-0.004	0
46	-0.01	0.01	0
47	0.005	-0.005	0
48	0.005	-0.005	0
49	0.011	-0.011	0
50	0.024	-0.024	0
51	-0.005	0.006	-0.001
52	0.007	-0.005	-0.002
53	0.004	-0.002	-0.002
54	-0.002	-0.011	0.013
55	-0.012	0.005	0.007
56	-0.005	-0.001	0.006
57	-0.005	0.013	-0.008
58	-0.002	0.01	-0.008
59	0.003	-0.002	-0.001
60	0.023	-0.014	-0.009
61	0.008	0.004	-0.012
62	0.002	-0.011	0.009
63	0.002	0.009	-0.011
64	-0.003	-0.001	0.004
65	0.006	-0.001	-0.005
66	-0.006	0.007	-0.001
67	-0.003	-0.002	0.005
68	-0.005	0.002	0.003
69	-0.011	0.01	0.001
70	0.004	-0.016	0.012

Appendix D

Estimation of MSE

In this section $x^{(b)}$ denotes the x value from the b^{th} iteration.

$$\begin{aligned}
M\hat{S}E(v_{ijt}) &= tr(\hat{\Sigma}) + bi\hat{a}s(v_{ijt})^T bi\hat{a}s(v_{ijt}) \\
&= \sum_{i,j} \frac{1}{n-1} \sum_b \left(v_{ijt}^{(b)} - \frac{1}{n} \sum_b v_{ijt}^{(b)} \right)^2 + \sum_{i,j} \left(\frac{1}{n} \sum_b v_{ijt}^{(b)} - n_{it} p_{ijt} \right)^2 \\
&= \sum_{i,j} \left[\frac{1}{n-1} \sum_b \left(v_{ijt}^{(b)} - \frac{1}{n} \sum_b v_{ijt}^{(b)} \right)^2 + \left(\frac{1}{n} \sum_b v_{ijt}^{(b)} - n_{it} p_{ijt} \right)^2 \right]
\end{aligned}$$

$$\begin{aligned}
M\hat{S}E(p_{ijt}) &= tr(\hat{\Sigma}) + bi\hat{a}s(p_{ijt})^T bi\hat{a}s(p_{ijt}) \\
&= \sum_{i,j} \frac{1}{n-1} \sum_b \left(p_{ijt}^{(b)} - \frac{1}{n} \sum_b p_{ijt}^{(b)} \right)^2 + \sum_{i,j} \left(\frac{1}{n} \sum_b p_{ijt}^{(b)} - p_{ijt} \right)^2 \\
&= \sum_{i,j} \left[\frac{1}{n-1} \sum_b \left(p_{ijt}^{(b)} - \frac{1}{n} \sum_b p_{ijt}^{(b)} \right)^2 + \left(\frac{1}{n} \sum_b p_{ijt}^{(b)} - p_{ijt} \right)^2 \right]
\end{aligned}$$

Appendix E

Summary of Simulation Outcomes in All Settings

Setting	γ	δ	ϕ	$\text{MSE}_{Uniform}/\text{MSE}_{Ratio}$ p_{ijt}	v_{ijt}	Recommend
1	0	0	0	0.988	0.988	Uniform
2	0	0.5	0	0.992	0.988	Uniform
2	0	1	0	0.990	0.989	Uniform
2	0	2	0	0.993	0.991	Uniform
2	0	5	0	0.998	0.996	Uniform
3	0	0	-0.3	0.985	1.000	Uniform
3	0	0	-0.05	0.989	0.996	Uniform
3	0	0	0.05	0.991	0.996	Uniform
3	0	0	0.3	0.991	1.000	Uniform
4	-10	0	0	0.910	0.958	Uniform
4	-1	0	0	0.962	0.976	Uniform
4	1	0	0	0.964	0.956	Uniform
4	10	0	0	0.911	0.903	Uniform
5	-10	0	-0.3	0.910	0.971	Uniform
5	-10	0	-0.05	0.910	0.955	Uniform
5	-10	0	0.05	0.911	0.963	Uniform
5	-10	0	0.3	0.910	0.988	Uniform
5	-1	0	-0.3	0.961	0.999	Uniform
5	-1	0	-0.05	0.963	0.989	Uniform
5	-1	0	0.05	0.963	0.990	Uniform
5	-1	0	0.3	0.964	1.000	Uniform
5	1	0	-0.3	0.961	0.999	Uniform
5	1	0	-0.05	0.964	0.980	Uniform
5	1	0	0.05	0.964	0.981	Uniform
5	1	0	0.3	0.964	0.999	Uniform

Setting	γ	δ	ϕ	$\text{MSE}_{Uniform}/\text{MSE}_{Ratio}$ p_{ijt}	v_{ijt}	Recommend
5	10	0	-0.3	0.911	0.951	Uniform
5	10	0	-0.05	0.912	0.902	Uniform
5	10	0	0.05	0.912	0.910	Uniform
5	10	0	0.3	0.912	0.960	Uniform
6	-10	0.5	0	0.908	0.955	Uniform
6	-10	1	0	0.908	0.952	Uniform
6	-10	2	0	0.907	0.946	Uniform
6	-10	5	0	0.910	0.930	Uniform
6	-1	0.5	0	0.962	0.976	Uniform
6	-1	1	0	0.967	0.977	Uniform
6	-1	2	0	0.973	0.979	Uniform
6	-1	5	0	0.990	0.989	Uniform
6	1	0.5	0	0.964	0.956	Uniform
6	1	1	0	0.965	0.957	Uniform
6	1	2	0	0.974	0.965	Uniform
6	1	5	0	0.990	0.983	Uniform
6	10	0.5	0	0.910	0.900	Uniform
6	10	1	0	0.909	0.898	Uniform
6	10	2	0	0.909	0.893	Uniform
6	10	5	0	0.914	0.880	Uniform
7	0	0.5	-0.3	0.988	1.000	Uniform
7	0	0.5	-0.05	0.990	0.996	Uniform
7	0	0.5	0.05	0.992	0.996	Uniform
7	0	0.5	0.3	0.991	1.000	Uniform
7	0	1	-0.3	0.986	1.000	Uniform
7	0	1	-0.05	0.990	0.996	Uniform
7	0	1	0.05	0.990	0.996	Uniform
7	0	1	0.3	0.992	1.000	Uniform
7	0	2	-0.3	0.991	1.000	Uniform
7	0	2	-0.05	0.993	0.996	Uniform
7	0	2	0.05	0.993	0.996	Uniform
7	0	2	0.3	0.994	1.000	Uniform
7	0	5	-0.3	0.997	1.000	Uniform
7	0	5	-0.05	0.998	0.997	Uniform
7	0	5	0.05	0.998	0.997	Uniform
7	0	5	0.3	0.999	1.000	Uniform
8	-10	0.5	-0.3	0.909	0.970	Uniform
8	-10	0.5	-0.05	0.909	0.952	Uniform
8	-10	0.5	0.05	0.909	0.960	Uniform
8	-10	0.5	0.3	0.909	0.986	Uniform
8	-10	1	-0.3	0.908	0.968	Uniform
8	-10	1	-0.05	0.908	0.949	Uniform
8	-10	1	0.05	0.908	0.957	Uniform
8	-10	1	0.3	0.908	0.985	Uniform

Setting	γ	δ	ϕ	$\text{MSE}_{Uniform}/\text{MSE}_{Ratio}$		Recommend
				p_{ijt}	v_{ijt}	
8	-10	2	-0.3	0.907	0.965	Uniform
8	-10	2	-0.05	0.906	0.943	Uniform
8	-10	2	0.05	0.907	0.952	Uniform
8	-10	2	0.3	0.907	0.983	Uniform
8	-10	5	-0.3	0.910	0.955	Uniform
8	-10	5	-0.05	0.910	0.927	Uniform
8	-10	5	0.05	0.911	0.936	Uniform
8	-10	5	0.3	0.910	0.974	Uniform
8	-1	0.5	-0.3	0.962	0.999	Uniform
8	-1	0.5	-0.05	0.963	0.988	Uniform
8	-1	0.5	0.05	0.963	0.989	Uniform
8	-1	0.5	0.3	0.963	1.000	Uniform
8	-1	1	-0.3	0.963	0.999	Uniform
8	-1	1	-0.05	0.967	0.988	Uniform
8	-1	1	0.05	0.966	0.989	Uniform
8	-1	1	0.3	0.967	1.000	Uniform
8	-1	2	-0.3	0.972	0.999	Uniform
8	-1	2	-0.05	0.972	0.989	Uniform
8	-1	2	0.05	0.971	0.988	Uniform
8	-1	2	0.3	0.974	0.999	Uniform
8	-1	5	-0.3	0.989	0.999	Uniform
8	-1	5	-0.05	0.990	0.991	Uniform
8	-1	5	0.05	0.990	0.991	Uniform
8	-1	5	0.3	0.990	0.999	Uniform
8	1	0.5	-0.3	0.962	0.999	Uniform
8	1	0.5	-0.05	0.964	0.980	Uniform
8	1	0.5	0.05	0.965	0.980	Uniform
8	1	0.5	0.3	0.965	0.999	Uniform
8	1	1	-0.3	0.964	0.999	Uniform
8	1	1	-0.05	0.966	0.980	Uniform
8	1	1	0.05	0.967	0.980	Uniform
8	1	1	0.3	0.967	0.999	Uniform
8	1	2	-0.3	0.971	0.999	Uniform
8	1	2	-0.05	0.973	0.981	Uniform
8	1	2	0.05	0.973	0.980	Uniform
8	1	2	0.3	0.974	0.999	Uniform
8	1	5	-0.3	0.989	0.999	Uniform
8	1	5	-0.05	0.990	0.988	Uniform
8	1	5	0.05	0.990	0.986	Uniform
8	1	5	0.3	0.990	0.999	Uniform
8	10	0.5	-0.3	0.910	0.950	Uniform
8	10	0.5	-0.05	0.910	0.900	Uniform
8	10	0.5	0.05	0.911	0.908	Uniform
8	10	0.5	0.3	0.911	0.959	Uniform

Setting	γ	δ	ϕ	$\text{MSE}_{Uniform}/\text{MSE}_{Ratio}$		Recommend
				p_{ijt}	v_{ijt}	
8	10	1	-0.3	0.909	0.949	Uniform
8	10	1	-0.05	0.910	0.897	Uniform
8	10	1	0.05	0.909	0.905	Uniform
8	10	1	0.3	0.909	0.958	Uniform
8	10	2	-0.3	0.908	0.946	Uniform
8	10	2	-0.05	0.908	0.892	Uniform
8	10	2	0.05	0.909	0.901	Uniform
8	10	2	0.3	0.909	0.955	Uniform
8	10	5	-0.3	0.913	0.936	Uniform
8	10	5	-0.05	0.914	0.879	Uniform
8	10	5	0.05	0.913	0.887	Uniform
8	10	5	0.3	0.913	0.947	Uniform