Delta Hedging for Single Premium Segregated Fund

by

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Abstract

Segregated funds are individual insurance contracts that offer growth potential of investment in underlying assets while providing a guarantee to protect part of the money invested. The guarantee can cause significant losses to the insurer which makes it essential for the insurer to hedge this risk. In this project, we discuss the hedging effectiveness of delta hedging by studying the distribution of hedging errors under different assumptions about the return on underlying assets. We consider a Geometric Brownian motion and a Regime Switching Lognormal to model equity returns and compare the hedging effectiveness when risk-free rates are constant or stochastic. Two one-factor short-rate models, the Vasicek and CIR models, are used to model the risk-free rate. We find that delta hedging is in general effective but large hedging errors can occur when the assumptions of the Black-Scholes’ framework are violated.

**Keywords:** Segregated Fund Guarantees; Delta Hedging; Regime Switching Model; Hedging Errors; Stochastic Modelling
Dedication

To my parents.
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Chapter 1

Introduction

1.1 Background and Motivation

Traditional life insurance products are mainly designed to provide benefits contingent on the death or survival of the insured life. As the insurance market advances, life insurance products are becoming more and more diversified. Policyholders have become more aware of investment opportunities within insurance contracts – bullish markets and low interest rate environments motivate investors to look for higher returns than those provided by conventional annuities. Insurers around the world have developed equity-linked insurance products to allow policyholders to enjoy both the investment return on the premium and a payment guarantee. This preserves the traditional insurance feature while partially passing the investment risk to policyholders.

In the late 1960s and early 1970s, unit-linked insurances, which combine a mutual fund type investment with a minimum payment guarantee, became very popular in the United Kingdom. In the United States, deferred annuities linked to an investment fund during the deferment period were introduced in the 1970s. Since the 1990s, certain payment guarantees have been incorporated in such products. These products are known as variable annuities. In Canada, insurers offer similar products with complex guaranteed values on death or maturity known as segregated funds. They became popular in the late 1990s.

There are different types of guarantees embedded in variable annuities known as guaranteed minimum benefit of type x (GMxB). A Guaranteed Minimum Maturity Benefit (GMMB) guarantees that the benefit at maturity will be the greater of the account value and some predefined minimum amount regardless of the investment performance, which acts like a downside protection for the policyholder’s fund. A Guaranteed Minimum Death Benefit (GMDB), guarantees that the beneficiary will receive at least a minimum amount upon the death of the policyholder. A Guaranteed Minimum Accumulation Benefit (GMAB)
guarantees that the policyholder’s contract value will be at least equal to a certain minimum percentage of the amount invested after a specified number of years. A GMAB policy may include multiple dates at which the fund may receive a top up if lower than the guaranteed value. The Guaranteed Minimum Income Benefit (GMIB) guarantees to pay at least a base amount of annual income for life regardless of the investment performance after the policyholder has annuitized the contract.

A typical segregated fund contract in Canada is a single premium policy with 100% of the premium being invested in mutual funds and kept in a separate account on behalf of the policyholder. The account value is protected by a GMMB and a GMDB. The insurer charges either a monthly management fee that is deducted from the fund value or an upfront management fee. Segregated funds have become increasingly popular over the past decade, as made evident by the following chart. The total assets at market value of segregated funds increased by 259% since 2001 and by 116% since 2008.

![Figure 1.1: Total assets at market value of segregated funds (in billions of Canadian dollars).](image)

Source: Statistic Canada

The segregated fund product is popular among insurance companies for several reasons. The first reason is regulatory concerns. The Client Relationship Model Phase 2 (CRM2) that was gradually phased-in from July 2013 to July 2016 requires advisors to disclose mutual fund fees and performance but does not mandate disclosure of fees or product performance for segregated funds. This has led financial advisers who hold both insurance and mutual fund licenses to favouring segregated funds over mutual funds and one can expect this trend to continue. Second, although insurance companies bear guarantee costs compared to mutual funds, it is possible for them to charge much higher management fees to cover the guarantee costs and earn profits. Management-expense ratios can be an additional 50 to
150 basis points on top of the cost of a mutual fund. With the average mutual fund MER being 2.4% in 2015, this adds up to a management-expense ratio of 2.9% to 3.9%. (O’Hara, 2015).

As for investors, apparently their main reason to favour segregated fund products is the guarantee. A 2014 survey conducted by Sun Life Financial suggests that 97% of respondents say it is important for them that some of their retirement income is guaranteed for the rest of their lives, which is up by 22% from 2008. (Taglioni, 2014). Creditor protection is another favourable feature of segregated funds. “When the contract’s named beneficiary is a spouse, child, grandchild or parent of the insured person or when the beneficiary is designated irrevocably or where the contract is registered (for example, as a Registered Retirement Savings Plan), creditors cannot seize a segregated fund contract if the insured person declares bankruptcy or fails to pay his or her debts, as long as the insured person has not entered into the contract for the primary purpose of shielding assets from creditors. Some contracts also permit the insured to the guarantee periodically in order to lock in increases in the market value of the segregated funds the contract has invested in.” (Canadian Life and Heath Insurance Association, 2014). In addition to this, the segregated fund usually provides probate protection – beneficiaries can receive their payout faster while the cost of probate fees is avoided. Lastly, the product is especially suited for aggressive investing where a guarantee on return is particularly attractive (Canadian Life and Heath Insurance Association, 2014).

However, segregated fund products do face some challenges. First, the market’s pressure on the high investment costs has become more and more prominent. This has led insurance companies to come up with low cost solutions, for example Empire Life Insurance’s 75/75 lower-cost option for investors, which provides a 75% maturity guarantee and 75% death benefit (O’Hara, 2015). Second, the Office of the Superintendent of Financial Institutions (OSFI) is designing a new regulatory framework known as “Life Insurance Capital Adequacy Test (LICAT)” to replace the current framework, “Minimum Continuing Capital and Surplus Requirements” (MCCSR) in January 2018. Compared to the factor-based MCCSR which oversimplifies the calculation of the required capital, LICAT is model-based and hence more rigorous and risk-sensitive. The insurance companies are then facing more regulatory supervision on the assessment of segregated fund risk, and will face more challenges in keeping this popular line of products profitable. (Kong, 2016).

Given these challenges, a few natural questions arise. First, how and to what extent the insurance company can make the segregated fund products profitable. Second, given the nature of this line of product – the downside protection of the fund investment – what are the most effective hedging strategies to prevent big losses when the equity market is performing poorly.
Generally speaking, three hedging methods are popular: dynamic hedging, static hedging and mixed hedging. This project focuses on dynamic hedging under stochastic interest rate and stochastic equity return using the real-world measure. We choose dynamic hedging over mixed hedging for its implementability, and over static hedging for its effectiveness. We use Vasicek and CIR models to generate stochastic interest rate. The Vasicek (1977) model is a popular one-factor short-rate model which has a mean-reverting feature. However, the Vasicek model assumes a constant local volatility, which might lead to negative interest rates. To overcome this disadvantage, one approach is to modify the process so that the local volatility tends to zero when the interest rate is very small. One of the models that possesses this feature is known as the Cox, Ingersoll and Ross (1985; henceforth CIR) model.

To generate equity returns, we use a geometric Brownian motion (GBM) and a regime-switching lognormal model (RSLN). The geometric Brownian motion is a standard model used in the insurance industry. However, in the real world, it has been observed that the volatility of equity market is higher in bear markets than in bull markets. The RSLN can help overcome this problem by assuming K regimes for which the data generating process switches randomly and each regime is a GBM with different mean and variance. The RSLN has the advantage of capturing the variable volatility while preserving the simplicity of the independent lognormal model.

1.2 Literature Review

Segregated fund products have drawn considerable attention in the academic world since the beginning of the 21st century. There is an extensive literature focusing on pricing, valuation and hedging of variable annuities.

Hardy (2000) compares three methods for determining appropriate provisions for maturity guarantees for single-premium segregated fund contracts: actuarial reserving, dynamic hedging and static hedging. Actuarial reserving assumes funds to be held in risk-free assets and reserves are determined so that the probability of the sum of the accumulated fund and the reserve being larger than the guarantee is bigger than a given standard. Dynamic hedging views the maturity guarantee as a put option which can be replicated by risky assets and bonds using the Black and Scholes framework. The static hedging methodology tries to replicate the maturity guarantee by purchasing external options. Coleman, Li and Patron (2006) consider quadratic risk minimization to hedge the options embedded in GMDBs with ratchet features under stochastic equity return and interest rate, which are modelled by a Merton (1973) jump diffusion model. The results suggest that an effective risk reduction can be achieved by using risk minimization hedging under a joint model for
the underlying and the interest rate. Coleman et al. (2007) consider a market model for volatility risks and then compute risk minimization hedging by modelling the at-the-money Black-Scholes implied volatility explicitly. The paper argues that risk minimization hedging using standard options is superior to delta hedging as delta hedging can lead to large hedging errors because it is sensitive to the presence of instantaneous volatility risk.

Instead of modelling equity return and interest rate under real-world pricing dynamic, some researchers use risk-neutral processes to do valuation and risk hedging. Milevsky and Salisbury (2006) compare a static model which assumes that the policyholders hold the product to maturity and a dynamic model which assumes that the policyholders lapse when there is an economic advantage to do so. The fund value is assumed to follow a geometric Brownian motion and the valuation is done using the risk-neutral measure, under which the real world drift is replaced by the risk-free rate. The main result is that the no-arbitrage hedging cost of a Guaranteed Minimum Withdrawal Benefit (GMWB) is actually much higher than the charges of most products in the market. Therefore, it is essential to increase the GMWB fees to avoid arbitrage opportunities. Bauer, Kling and Russ (2008) propose a general framework that can be used to price all types of guarantees currently offered within variable annuity contracts consistently. The fund value is assumed to follow a geometric Brownian motion and the financial market is assumed to be independent from biometric events. By using Monte Carlo simulation and a multidimensional discretization approach the paper concludes that the GMDB is overpriced while the prices of other guarantees such as the GMIB are significantly lower than the risk-neutral value. Wang (2009) proposes quantile hedging for GMDB under the risk-neutral framework: a geometric Brownian motion is used to model the equity return and the risk-free rate is assumed to be constant. The main contributions of this paper are the formulation of the quantile hedging problem for GMDB, the rigorous analysis, the closed-form solutions for special cases, and a framework for numerical solutions in general. Bacinello et al. (2011) discuss the valuation of GMDB, GMIB, GMAB and GMWB. They compute and compare contract values and fair fee rates under static and mixed valuation approaches using ordinary and least squares Monte Carlo methods, respectively. Their main conclusion is that a progressive shift from deterministic methods to stochastic approaches is apparent and applications of stochastic models in life insurance have become increasingly popular due to the awareness of financial risks and of biometric systematic risks (the aggregate longevity risk in particular).

Holz, Kling and Ruß (2012) use risk-neutral measures to price guaranteed lifelong withdrawal benefits (GLWB) when the policyholder behaviours are deterministic, probabilistic and stochastic. The fund value is assumed to follow a geometric Brownian motion. The fair guarantee fees are calculated for different types of contracts and sensitivity testing is conducted with respect to guaranteed annual withdrawal amount, insured’s age, mortality assumptions (i.e., different mortality table) and capital market parameters. This paper
shows that although GLWB products are less sensitive to policyholder behaviour than other guarantees typically embedded in variable annuities once withdrawals have started, the sensitivity with respect to changes in interest rates and volatilities as well as mortality rates is significant. This can be very risky considering the long time horizon of segregated funds. Feng and Volkmer (2012) compare the accuracy and efficiency of three different risk measure methods for GMDB and GMMB: direct calculation, numerical inversion of Laplace transform and Monte Carlo simulations. The analytical methods are shown through numerical experiments to be far more efficient and accurate than the Monte Carlo simulations. Instead of assuming a constant risk-free rate, there is also literature that considers modelling stochastic interest rate. Unlike Milevsky and Salisbury (2006) which assumes a constant risk-free rate, Peng, Leung and Kwok (2012) consider the pricing of variable annuities with a Guaranteed Minimum Withdrawal Benefit (GMWB) under a Vasicek stochastic interest rate framework. The paper also considers the correlation between equity return and interest rate. Under the assumption of deterministic withdrawal rates, the paper derives analytic approximation solutions to calculate the price of the GMWB under both equity and interest rate risks and obtains both the lower and upper bounds from the price functions. They find the lower bound and upper bound on the put option values using Monte Carlo simulations. After comparing the numerical accuracy of these two methods, Monte Carlo simulation is found to be sufficiently accurate even under long maturity and high volatility.

There is literature analyzing more than two variable annuity riders, for example, Sun (2006) considers pricing and risk management for variable annuity products that have various combinations of Guaranteed Minimum Benefit (GMB) riders. The paper measures profitability as the present value of distributable profits and stability of earnings as the volatility of profitability. The results show that, generally speaking, profitability is improved by adding more GMB riders to the existing variable annuity products (with or without GMB riders). However, the stability of earnings is deteriorating at the same time, which is consistent with the rule that the higher the return, the higher the risk. In terms of risk management, the paper also suggests that the volatility of earnings on variable annuities can be offset with options or futures trading in the capital market. The asset dynamic is assumed to follow a regime-switching lognormal process.

Some literature focuses on hedging the risks of the guarantees. Kling, Ruez and Russ (2011) consider different dynamic hedging strategies (for example, delta, rho and vega hedging) for a GLWB with different types of ratchet. The results show that the insurer’s risk changes dramatically when stochastic volatility is introduced and the risk can be significantly reduced when suitable hedging strategies are implemented. Kling, Ruez and Russ (2014) analyze the impact of policyholders’ behaviour on the hedging efficiency. They find that the insurer’s risk heavily depends on the difference between assumed and actual policyholder behaviour. Augustyniak and Boudreault (2012) use 78 different econometric models
including GARCH and regime-switching models to analyze the insurer’s risk due to the investment guarantee offered in equity-linked products. The paper also implements the Black and Scholes delta hedging strategy and the results show that hedging errors can be very large which suggest that model risk should be taken into consideration when hedging the guarantees.

Pricing and hedging options and general contingent claims are also discussed in the literature. Engelmann et al. (2006) provide an empirical comparison of the hedging performance of static hedging with that of dynamic hedging for barrier options. They find that static hedges can perform significantly better than dynamic hedges even when dynamic strategies hedge higher-order greeks. El Karoui and Quenez (1995) study the problems of pricing contingent claims or options in an incomplete market where prices cannot be derived based on no-arbitrage theory. The result is that the maximum price is the smallest price that allows the seller to hedge completely by a controlled portfolio of the basic securities. Schweizer (1992) solves the continuous-time hedging problem with a mean-variance objective for general contingent claims. The paper uses exponential Brownian motion to price assets when the rates of return between assets are assumed to be correlated.

1.3 Outline

This project focuses on one of the dynamic hedging methods – delta hedging for segregated fund guarantee. Different stochastic models were used to model the stock index and interest rate. Based on the delta hedging and these models, the distribution of hedging error is studied.

The project is organized as follows: Chapter 2 describes the technical details of segregated funds and delta hedging. Numerical results of delta hedging under a Black and Scholes framework and a constant risk-free rate for maturity guarantees are reported. Chapter 3 introduces two stochastic interest rate models – the Vasicek model and the CIR model. Discretization methods of these two models are addressed and simulation is done to study the paths of interest rate for different mean-reverting speed parameters. Chapter 4 studies the distribution of the hedging error assuming an annual fee rate is charged. The numerical results of the Vasicek model and the CIR model are compared. Chapter 5 introduces the regime-switching model and the maximum likelihood estimation method for the two-state regime switching lognormal model (RSLN-2). Historical stock index returns are fitted to the RSLN-2 model and new stock paths are simulated using the estimated parameters. The final chapter provides some conclusions.
Chapter 2

Delta Hedging Under the Geometric Brownian Motion Model

2.1 Segregated Fund

Segregated funds are distinct types of investment products sold by life insurance companies. They are individual contracts that invest in one or more underlying assets, such as mutual funds. Similar to a mutual fund, a segregated fund is a professionally managed pool of money that invests in different assets including stocks, bonds or other securities. However, unlike a mutual fund, a segregated fund is an insurance product, thus embedding special benefits. Policyholders of segregated fund products (and their beneficiaries) enjoy protection of their initial investment through maturity and death benefit guarantees: when the policy matures or the policyholders pass away, at least a specified percentage, commonly ranging from 75% to 100%, of the original value will be paid to policyholders or their beneficiaries. To benefit from the guarantee, however, policyholders have to hold their investment for a certain length of time (usually 10 years). Once a segregated fund is established, the policyholder is free to switch from one fund to another and he or she can choose to make additional lump sum or regular payments. The policyholder can withdraw money if he or she wants to, but withdrawals affect the maturity and death guarantees. In addition to protecting the policyholders’ investment when the market performs poorly, a segregated fund has other benefits:

- Protection from creditors: money invested in segregated funds may also be protected against seizure by creditors. This can be an important advantage for business owners and professionals who want to protect against an unexpected lawsuit or bankruptcy.
A way to pass on wealth: segregated fund contracts purchased with non-registered money let policyholders name beneficiaries, so the death benefit bypasses their estate and goes directly to the beneficiaries. Contract holders can also control how they get the benefit: as a lump sum or in the form of a payout annuity (Sun Life Financial, 2017). A typical segregated fund contract has the following features:

1. Insurance contracts: segregated funds are sold by life insurance companies as deferred annuity contracts and must be kept separate from the insurer’s assets.
2. Maturity dates: all segregated fund contracts have maturity dates and the holding periods to reach maturity are usually 10 or more years.
3. Maturity and death guarantee: There are two different types of guarantees within segregated fund contracts - a guaranteed minimum maturity benefit (GMMB) and a guaranteed minimum death benefit (GMDB). A GMMB secures that the policyholder receives the greater of the accumulated fund value and the guarantee if he or she survives to the maturity of the contract. A GMDB pays the beneficiary the maximum between the accumulated fund value and the guarantee upon death of the policyholder.
4. Management expense ratio (MER): segregated fund contracts usually charge management fees as a percentage of the investor’s account value. The management fee is usually higher than that of a mutual fund so as to cover the cost of the insurance features.
5. Reset option: segregated fund contracts usually have a reset option that allows the policyholder to lock in investment gains if the market value of the fund increases. This resets the contract’s deposit value to the greater of the deposit value and current market value, restarts the contract term, and extends the maturity date.

The shorter the term of the maturity guarantees on investment funds, the higher the risk exposure of the insurer. This inverse relationship is based on the premise that there is a greater chance of market decline over shorter periods. A contract holder’s use of reset provisions also contributes to costs, since resetting the guaranteed amount at a higher level means that the issuer will be liable for a higher amount.

2.2 Notation

In this project, we focus on a single premium segregated fund contract with GMMB that matures in 10 years and has no reset. The notation we use is set as follows:

- $F_t$: the market value of the segregated fund at time $t$, assuming the policy is still fully in force;
- $S_t$: the value of the underlying equity index at time $t$;
- $G$: the guarantee of the contract;
- $T$: the maturity date;
• $m$: the per-period MER deducted from the investor’s account.

The relationship between the fund value and the underlying equity index can be written as:

$$F_t = (1 - m)^t F_0 \frac{S_t}{S_0}. \quad (2.1)$$

### 2.3 Black-Scholes Framework and Delta Hedging

The guarantee of a segregated fund can be regarded as a put option with the strike price equal to the guarantee value. One of the most famous models used to price European options is the Black and Scholes (1973, henceforth B-S) framework. Under the B-S framework, the price of a risky asset $S_t$ is assumed to follow a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (2.2)$$

where

• $\mu$ is the annualized mean of the instantaneous return;
• $\sigma$ is the annualized standard deviation of the instantaneous return;
• $W_t$ is a Wiener process.

The solution to the above stochastic differential equation can be written as:

$$S_t = S_0 \exp \left( (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right). \quad (2.3)$$

**Proposition 2.1.** Over a horizon of $t$, the price of the risky asset under the geometric Brownian motion (2.3) has a lognormal distribution with parameters $(\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t)$ and $\sigma\sqrt{t}$.

**Proof:**

$$F_X(x) = \mathbb{P}[X \leq x]$$

$$= \mathbb{P}[S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t) \leq x]$$

$$= \mathbb{P}[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \leq \ln(x/S_0)]$$

$$= \mathbb{P}[W_t \leq (\ln(x/S_0) - (\mu - \frac{1}{2}\sigma^2)t)/\sigma]$$

$$= \mathbb{P}[W_t/\sqrt{t} \leq (\ln(x/S_0) - (\mu - \frac{1}{2}\sigma^2)t)/(\sigma\sqrt{t})]$$

$$= \int_{-\infty}^{(\ln(x/S_0) - (\mu - \frac{1}{2}\sigma^2)t)/(\sigma\sqrt{t})} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)dy. \quad (2.4)$$
Differentiating with respect to $x$ leads to the probability density function:

$$
f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left( -\frac{\ln(x) - \ln(S_0) - (\mu - \frac{1}{2}\sigma^2)t}{2\sigma^2 t} \right). \quad (2.5)$$

From Proposition 2.1 we can obtain the expected value and variance of $S(t)$:

$$E[S_t] = S_0 \exp((\mu - 0.5\sigma^2)t + 0.5\sigma^2 t) = S_0 \exp(\mu t) \quad (2.6a)$$

$$Var[S_t] = S_0^2 \exp(2(\mu - 0.5\sigma^2)t + 2\sigma^2 t) - S_0^2 \exp(2\mu t)$$

$$= S_0^2(\exp(2\mu t + \sigma^2 t) - \exp(2\mu t)). \quad (2.6b)$$

Under the Black and Scholes framework, there are some additional assumptions:

- the risk-free interest rate and the volatility of the underlying asset are known and stay constant through time;
- the stock does not pay any dividends;
- market is assumed to be frictionless, which means no transaction fees or taxes and trading is continuous;
- short selling is allowed, and there are no restrictions on short selling.

To derive the Black-Scholes formula, let us first derive the famous Black and Scholes differential equation (Hull, 2008). Let $V \equiv V(S,t)$ denote the value of a derivative on the stock. By Itô’s lemma, we have

$$dV = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t. \quad (2.7)$$

Let us consider constructing a portfolio that contains one derivative contract and $\Delta = -\frac{\partial V}{\partial S}$ shares of the stock, then the value of the portfolio becomes:

$$P = V - \frac{\partial V}{\partial S} S. \quad (2.8)$$

The stochastic differential equation can be written as follows:

$$dP = \left( \frac{\partial P}{\partial t} + \mu S \frac{\partial P}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \right) dt + \sigma S \frac{\partial P}{\partial S} dW_t$$

$$= \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} - \frac{\partial V}{\partial S} \right) dt + \sigma S \left( \frac{\partial V}{\partial S} - \frac{\partial V}{\partial S} \right) dW_t$$

$$= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.9)$$
The instantaneous uncertainty of the portfolio is zero, that is the change in the value of the portfolio depends only on time and has no stochastic term. Hence, the portfolio is instantaneously riskless and therefore, using a no-arbitrage argument, it must earn the risk-free rate:

\[
\begin{align*}
\frac{dP}{dt} &= rPdt \\
&= r \left( V - \frac{\partial V}{\partial S} S \right) dt.
\end{align*}
\] (2.10)

Equations (2.9) and (2.10) together lead to the following partial differential equation

\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV,
\] (2.11)

which is a key to deriving the closed-form solution for the Black and Scholes equation.

For such equation to have a unique solution, we will need boundary conditions on \( V(S, t) \). Subject to the terminal payoff of the derivative (for example, \( V_T = \max(K - S_T, 0) \) for a European put option), the analytical formulas for call and put options are derived. The Black and Scholes formula to value a put option at time \( t \) can be written as:

\[
P_t = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1),
\] (2.12)

where

\[
\begin{align*}
d_1 &= \frac{\ln(S_t/K) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (2.13a) \\
d_2 &= \frac{\ln(S_t/K) + (r - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \quad (2.13b)
\end{align*}
\]

Following Hardy (2003), the delta for a European put option at time \( t \) can be derived as follows:

\[
\Delta_P(t) = \frac{\partial P_t}{\partial S_t} = Ke^{-r(T-t)}N'(-d_2) - S_tN'(-d_1) - N(-d_1),
\] (2.14)

since

\[
d_2 = d_1 - \sigma \sqrt{T-t}, \quad (2.15)
\]
we have
\[ d_2^2 = d_1^2 - \sigma^2(T - t) - 2d_1 \sigma \sqrt{T-t} \]
\[ = d_1^2 + \sigma^2(T - t) - 2 \left( \ln S_t - \ln K + (r - \frac{1}{2} \sigma^2)(T - t) \right) \]
\[ = d_1^2 - 2 \ln S_t + 2 \ln K - 2r(T - t) \]
\[ \frac{d_2^2}{2} = \frac{d_1^2}{2} - \ln S_t + \ln K - r(T - t). \]  
(2.16a)

Now,
\[ N'(-d_1) = N' \left( \frac{-\left( \ln S_t - \ln K + (r - \frac{1}{2} \sigma^2)(T - t) \right)}{\sigma \sqrt{T-t}} \right) \]
\[ = -\frac{e^{-d_1^2/2}}{S_t \sigma \sqrt{2\pi(T-t)}}, \]  
(2.17)
and similarly
\[ N'(-d_2) = -\frac{e^{-d_2^2/2}}{S_t \sigma \sqrt{2\pi(T-t)}}. \]  
(2.18)

Therefore,
\[ Ke^{-r(T-t)}N'(-d_2) = -Ke^{-r(T-t)} \frac{e^{-d_2^2/2}}{S_t \sigma \sqrt{2\pi(T-t)}} \]
\[ = -\frac{Ke^{-r(T-t)-(d_2^2/2-\ln S_t+\ln K-r(T-t))}}{S_t \sigma \sqrt{2\pi(T-t)}} \]
\[ = -\frac{e^{-d_2^2/2+\ln S_t}}{S_t \sigma \sqrt{2\pi(T-t)}} \]
\[ = -\frac{e^{-d_2^2/2}}{S_t \sigma \sqrt{2\pi(T-t)}} \]
\[ = -S_t N'(-d_1). \]  
(2.19)

Thus we have
\[ \Delta_P(t) = \frac{\partial P_t}{\partial S_t} = -N(-d_1). \]  
(2.20)

Using a delta hedging strategy to hedge the risk of the put option, one should hold \( \Delta_P(t) \) shares of risky assets and \( P_t - \Delta_P(t) \cdot S_t \) in a risk-free money-market account at time \( t \). This hedging portfolio replicates the payoff of the option and is self-financing. It means that in each infinitesimal time step, the change in the amount needed in risky assets is exactly offset by an opposite change in the amount needed in the money-market account.
Unfortunately, delta is a variable of time, and in order to maintain a completely delta-hedged portfolio, one must rebalance the portfolio by making sure that at every infinitesimal time step $dt$ the portfolio contains delta shares of the underlying asset. In practice, however, one cannot trade continuously. During the time gaps when the hedge is not adjusted, it is highly possible for the change in the stock part to be different from the change in the bond part, which results in hedging errors.

Suppose we want to hedge a European option using a portfolio of bonds and stocks on a weekly basis. A delta hedge strategy requires the portfolio to be insensitive to small changes in stock price, that is

$$\Delta P(t) = \frac{\partial P}{\partial S} = 0, \quad t = 0, \frac{1}{52}, \frac{2}{52}, \ldots, 1, \frac{1}{52}, \ldots, 10.$$ \hspace{1cm} (2.21)

Then the value of the hedging portfolio can be written as:

$$P(t) = B_t + \Delta P(t) \cdot S_t$$ \hspace{1cm} (2.22)

where $\Delta P(t) \cdot S_t$ is the amount of dollars invested in the stock, and $B_t$ is the number of units of the bond. Right before rebalancing at time $t$, the value of the hedge portfolio set up at time $t - \frac{1}{52}$ has accumulated to

$$P(t^-) = B_{t-\frac{1}{52}} e^{r/52} + \Delta P(t - \frac{1}{52}) \cdot S_t = B_t + \Delta P(t - \frac{1}{52}) \cdot S_t, \quad t = \frac{1}{52}, \frac{2}{52}, \ldots, 1, \frac{1}{52}, \ldots, 10.$$ \hspace{1cm} (2.23)

The difference $P(t) - P(t^-)$ is the hedging error in a period. It is obvious that the longer the time step, the larger the hedging error. Choosing the frequency of hedging is a trade-off between transaction costs and accuracy.

### 2.4 Numerical Results

In a 10-year GMMB with weekly rebalancing, under which a lump-sum premium is paid by the insured at the beginning and is invested in a general index fund tracking an appropriate stock index, the segregated fund provides a guarantee that is a percentage of the total gross premium before any deduction of management charges. The distribution of the cost of the guarantee is highly skewed since the payoff is zero under most cases, but can be significant if the fund performs poorly. The GMMB can be considered as a put option with a strike price equal to the guaranteed amount.

Let us now consider a 10-year GMMB with 100% guarantee. The single premium is 100 dollars. In this section, we assume a constant risk-free rate of 2% and a 10% upfront
management fee charged at the beginning of the term. Therefore, an upfront fee of 10 dollars is deducted from the investor’s account at the beginning and the actual initial fund value invested, $F_0$, becomes 90 dollars while the guaranteed amount, $G$, is still 100 dollars. The underlying stock index follows a geometric Brownian motion, i.e.,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \quad (2.24)$$

with parameters:
- $S_0 = 100$ (starting price of $S_t$);
- $\mu = 0.06$ (long-term mean of equity return);
- $\sigma = 0.2$ (volatility of equity return).

Note that the 10% upfront fee does not cover the full cost of the guarantee. A fee of 10% was chosen because it represents the sum of the annual management fees of 1% of the fund value used in Chapter 4. The value of the guarantee which is given by (2.12) is $P(0) = 17.4937$. Our simulation results ignore the shortfall between $P(0)$ and the $10$ fee, therefore overestimating the realized profits when the insurer hedges the guarantee.

Here, 10,000 paths of log-returns are simulated to project paths of the stock price over the next ten years at a weekly frequency. Figure 2.1 shows the projected paths of the stock price, from which we can tell that the stock index with initial value of 100 can reach 1600 after 10 years, but the stock price rarely goes beyond 1000 (only 6 paths of stock price go beyond 1000 in year ten). Figure 2.2 shows the distribution of $S_{10}/S_0$, which confirms that the distribution of the stock index is highly skewed and is an approximate lognormal distribution. Table 2.1 compares the theoretical mean and variance with the simulation results and we can conclude that they are reasonable.

Now since we have values of the stock index over the 10-year period, we can calculate a kind of profit or loss for the insurer at maturity (denoted by $\Pi$). Without hedging the guarantee, we define $\Pi$ as

$$\Pi = -\max(G - F_{10}, 0) + 0.1 \cdot S_0, \quad (2.25)$$

which is a negative cash flow equal to the payoff of the guarantee plus the upfront management fee without interest.

Figure 2.3 shows the distribution of these profits and losses of the insurer and we can see that the chances that there is no cost to the insurer are higher but it is also possible that the insurer will suffer from large losses. It is therefore essential for the insurer to hedge the financial risk of the guarantee.
To reduce the risk of having great losses, the fund manager can use the underlying equity and bonds to delta hedge the position. Using the Black and Scholes framework, one is able to price the GMMB and delta-hedge it by rebalancing the portfolio on a weekly basis.

When delta-hedging the guarantee at a weekly frequency, we define \( \Pi \) as the cumulative hedging errors less the guarantee payoff and plus the upfront management fee. That is

\[
\Pi = \sum_{t \in \tau} (P(t) - P(t^-)) e^{r(10-t)} - \max(G - F_{10}, 0) + 0.1 \times S_0 \quad (2.26)
\]

where \( \tau = \{ \frac{1}{52}, \frac{2}{52}, \ldots, 1, \frac{11}{52}, \ldots, 10 \} \).

Figure 2.1: Simulated paths of stock price under GBM using \( \mu = 0.06 \) and \( \sigma = 0.2 \).
Figure 2.2: Distribution of $S_{10}/S_0$ under GBM using $\mu = 0.06$ and $\sigma = 0.2$.

Table 2.1: Stock price after 10 years

<table>
<thead>
<tr>
<th></th>
<th>Expected value of $S_{10}$</th>
<th>Standard deviation of $S_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical Value</td>
<td>182.21</td>
<td>127.78</td>
</tr>
<tr>
<td>Simulated Value</td>
<td>181.76</td>
<td>126.40</td>
</tr>
</tbody>
</table>

Using Equation (2.20), the delta of the guarantee at time 0 is $\Delta_P(0) = -0.3207$. In order to hedge the guarantee, at time 0, the insurer sells $28.8590$ of stocks and spends $46.3527$ on bonds. After 10 years, the fund value is $0.9 \times S_{10}$.

Figure 2.3: Distribution of profits and losses without hedging the guarantee.

Figure 2.4: Distribution of profits and losses with hedging the guarantee.
Delta hedging is implemented using the 10,000 simulated stock paths and hedging errors are tracked during the whole 10 years for each path.

The resulting approximate distribution of the cumulative hedging error at maturity and paths of the hedging error over time are illustrated in Figures 2.5 and 2.6, respectively. They show that with weekly rebalancing, the variance of the delta hedging error is very small relative to the fund value. Figure 2.4 shows the distribution of profits and losses using delta hedging with weekly rebalancing. Similarly, the 10 dollars of management fee is added back directly without considering the time value of money. By comparing these two distributions of profit, one can conclude that the distribution of profit when hedging the guarantee is not highly skewed anymore. In fact, it looks more like a normal distribution with a mean around 10. Also, delta hedging substantially reduces the risk of large losses for the insurer.

In the next three chapters, we relax the interest rate assumption by making it stochastic. However, the fund manager is still assumed to price the guarantee and delta hedge it under the Black and Scholes framework. That is, while the fund manager uses each period’s interest rate to reprice the guarantee, she assumes the interest rate will be constant from this period onward.
Chapter 3

Delta Hedging Under Stochastic Interest Rate

In the previous chapter, the risk-free rate is assumed to be constant, which is acceptable when the risk-free rate is not the dominant state variable that determines the option payoff, and when the life of the option is relatively short (Bakshi, Cao and Chen, 1997). However, since the life span of a segregated fund is usually 10 years or even longer, it is unrealistic to make such assumption for such a contract. Although delta hedging under a constant risk-free rate is highly effective, we want to see whether this hedging method still works well when the interest rate is actually stochastic. In this chapter, we assume the fund manager still delta hedges the guarantee using the Black and Scholes framework, but the interest rate is no longer constant within different time periods. Two popular interest rate models are introduced and numerical results are reported.

3.1 Vasicek Model

Mean-reverting processes are widely used in finance and frameworks that have this characteristic are used to model the term structure of interest rate. One of the popular models is the Vasicek (1977) model – based upon the Ornstein-Uhlenbeck (OU) process – which is one of the simplest one-factor short-rate models. The process is defined as:

\[ dr_t = \alpha(\mu - r_t)dt + \sigma dB_t \]  

(3.1)

where

- \( \mu \) represents the long-term mean risk-free rate;
- \( \alpha \) represents the mean-reverting rate;
• $\sigma$ represents the local volatility of the risk-free rate;
• $B_t$ is a standard Brownian motion;
• $\mu, \alpha, \sigma$ are all positive constants.

Under the Vasicek model, the interest rate $r_t$ reverts to the unconditional mean $\mu$ and $\alpha$ measures the reverting speed. When $\alpha$ is large, $r_t$ goes back to $\mu$ very quickly. Mean-reverting processes are attractive when it comes to modelling interest rates because they embody the economic argument that when the interest rates are high, the economy tends to slow down and borrowers tend to borrow less. Hence, the rates are pulled back to $\mu$. When the interest rates are low, demand for funds tends to be high, leading the rates to increase. Without this feature, interest rates could drift permanently upward the same way stock prices do and this is simply not observed in practice. Figure 3.1 illustrates the main features of Ornstein-Uhlenbeck process:

Figure 3.1: Ornstein-Uhlenbeck process for interest rate

Another feature of the model is that there is an analytical solution to $r_{t+s}$ given $r_t$ for any $s > 0$. The solution of $r_{t+s}$ given $r_t$ can be written as:

$$r_{t+s} = r_t e^{-\alpha s} + \mu(1 - e^{-\alpha s}) + \sigma \int_t^{t+s} e^{-\alpha s} dB_t.$$  \hspace{1cm} (3.2)

Therefore, $r_{t+s}$ given $r_t$ is normally distributed with mean:

$$E[r_{t+s}|r_t] = \mu + (r_t - \mu)e^{-\alpha s}$$  \hspace{1cm} (3.3)

and variance

$$\text{Var}[r_{t+s}|r_t] = \sigma^2[1 - e^{-2\alpha s}]/2\alpha.$$  \hspace{1cm} (3.4)
The long-term unconditional mean and variance of \( r_t \) are \( \mu \) and \( \sigma^2 / 2\alpha \), respectively.

Under the Vasicek model, there is also an analytical solution for the price, at time \( t \), of a zero-coupon bond maturing at \( T \) with face value of one dollar (see, for example, Cairns, 2004, p.65). This price is

\[
P(t, T) = e^{A_V(T-t)-B_V(T-t)r_t}
\]

where

\[
B_V(T - t) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}
\]

and

\[
A_V(T - t) = (B_V(T - t) - (T - t)) \left( \mu - \frac{\sigma^2}{2\alpha^2} \right) - \frac{\sigma^2}{4\alpha} B_V(T - t)^2.
\]

Yet, the Vasicek model has weaknesses. For this model, the local volatility \( \sigma \) is constant, which is inconsistent with what we observe in the real world. Also, when \( r_t \) is very small, it is possible for \( r_t \) to become negative.

### 3.2 CIR Model

To overcome the disadvantage of generating negative interest rates under the Vasicek model, one approach is to model the logarithm of \( r_t \), so a \( \ln r_t \) of below zero still corresponds to a spot rate above zero. Another approach is to modify the OU process so that the local volatility tends to zero when \( r_t \) is very small. One of the models that has this characteristic is the Cox, Ingersoll and Ross (1985) model and the dynamic process for \( r_t \) is defined as:

\[
dr_t = \alpha(\mu - r_t)dt + \sigma\sqrt{r_t}dB_t
\]

where \( \mu, \alpha, \sigma \) are all positive constants and \( B_t \) is a standard Brownian motion. Some desirable features of the CIR model include positive \( r_t \) as long as \( 2\alpha\mu \geq \sigma^2 \) and interest rate-dependent volatility. When the interest rate is very low, the local volatility gets close to zero. Then the drift term becomes the dominant factor that drives the interest rate back towards its long-term mean.

Under the CIR model, we can show that, for \( s > 0 \), \( r_{t+s} \) given \( r_t \) has expected value \( \mu + (r_t - \mu)e^{-\alpha s} \) and variance

\[
\frac{\sigma^2}{\alpha} (e^{-\alpha s} - e^{-2\alpha s}) + \frac{\mu\sigma^2}{2\alpha} (1 - e^{-2\alpha s})^2.
\]

The long-term unconditional variance of \( r_t \) is \( \mu\sigma^2 / 2\alpha \). The analytical solution for the price, at time \( t \), of a zero-coupon bond maturing at \( T \) with face value of one dollar (see, for
example, Cairns, 2004, p.67) can be written as:

\[ P(t, T) = e^{A_C(T-t) - B_C(T-t) r_t} \quad (3.9) \]

where

\[ B_C(\tau) = \frac{2(e^{\gamma \tau} - 1)}{(\gamma + \alpha)(e^{\gamma \tau} - 1) + 2\gamma} \quad (3.10) \]

and

\[ A_C(\tau) = \frac{2\alpha \mu}{\sigma^2} \ln \left( \frac{2\gamma e^{(\gamma + \alpha)\tau/2}}{(\gamma + \alpha)(e^{\gamma \tau} - 1) + 2\gamma} \right). \quad (3.11) \]

In general, the CIR model solves the main issue of the Vasicek model, while allowing for an analytical pricing formula. In addition, the feature of an interest rate-dependent volatility in the CIR model is consistent with empirical evidence that the volatility is relatively higher when interest rate is high.

As we assume that the fund manager delta hedges the guarantee under the Black and Scholes framework, the bond pricing formulas (3.5)–(3.7) and (3.9)–(3.11) are not used. While the interest rate is stochastic, we assume that it is constant within each period, so we still can use (2.23) to obtain the accumulated value of the hedging portfolio right before rebalancing. Then Equation (2.23) can be rewritten as

\[ P(t-\frac{1}{52}) = B_{t-\frac{1}{52}} e^{r_{t-\frac{1}{52}} - \frac{1}{52}} + \Delta P(t - \frac{1}{52}) \cdot S_t, \quad t = \frac{1}{52}, \frac{2}{52}, \ldots, 1, \frac{1}{52}, \ldots, 10. \quad (3.12) \]

The difference \( P(t) - P(t^-) \) is the hedging error in period \( t \).

### 3.3 Discretization of Vasicek and CIR Models

Numerical simulations of the Vasicek and CIR models usually require discretization, which means calculating samples at discrete time steps of length \( \Delta t \).

For the Vasicek model, one may write the discretization as:

\[ r_t = r_{t-1} + \alpha(\mu - r_{t-1})\Delta t + \sigma Z_t \quad (3.13) \]

where \( Z_t \sim N(0, 1) \). But this simulation approach is only valid when the time step, \( \Delta t \), is sufficiently small. It is better to use the covariance equivalence principle for discretizing the process; that is, to assume that the Vasicek model and its discretization have the same mean and stationary variance at all times. Pandit and Wu (1983) consider a uniform sampling of a continuous first-order autoregressive system, denoted by AR(1). Consider an AR(1)
model that has the form:

\[ X_t = \phi X_{t-1} + a_t, \quad a_t \sim N(0, \sigma_a^2). \quad (3.14) \]

If the AR(1) model has a positive autoregressive parameter, then it can represent a sampled first-order continuous system. The corresponding parameters of the AR(1) model are simply obtained from

\[ \phi = e^{-\alpha \Delta t} \quad (3.15) \]

and

\[ \sigma_a^2 = \frac{\sigma^2 (1 - \phi^2)}{2\alpha}. \quad (3.16) \]

Therefore a new discretization formula that has the same mean and stationary variance as the continuous process for any given \(\Delta t\) can be written as the following AR(1) model:

\[ r_t = e^{-\alpha \Delta t} r_{t-1} + (1 - e^{-\alpha \Delta t}) \mu + \sigma \sqrt{\frac{1 - e^{-2\alpha \Delta t}}{2\alpha}} Z_t, \quad (3.17) \]

where \(Z_t \sim N(0,1)\).

To construct a discretization for the CIR model, one can use the Euler-Maruyama scheme. For a time step of \(\Delta t\), the approximating process is as follow:

\[ r_t = r_{t-1} + \alpha (\mu - r_{t-1}) \Delta t + \sigma \sqrt{\Delta t} \left| r_{t-1} \right| Z_t, \quad (3.18) \]

where \(Z_t \sim N(0,1)\). It gives appropriate values of \(r_t\) when \(r_{t-1}\) is non-negative. However this approach can be problematic if \(r_{t-1}\) happens to be negative since we are taking the square root of \(r_{t-1}\). To avoid this problem, one may simulate values of \(Z_t\) until \(r_t\) is nonnegative, but this method raises another problem. The number of steps needed to generate a sample of any given size becomes random. Some other ways, listed in Labbé et al. (2010), deal with the negative \(r_t\):

\[
\begin{align*}
    r_t &= r_{t-1} + \alpha (\mu - r_{t-1}) \Delta t + \sigma \sqrt{\Delta t} \sqrt{r_{t-1}^{+}} Z_t \quad (3.19a) \\
    r_t &= r_{t-1} + \alpha (\mu - r_{t-1}^{+}) \Delta t + \sigma \sqrt{\Delta t} \sqrt{r_{t-1}^{+}} Z_t \quad (3.19b) \\
    r_t &= \left| r_{t-1} + \alpha (\mu - r_{t-1}) \Delta t + \sigma \sqrt{\Delta t} \sqrt{r_{t-1}^{+}} Z_t \right| \quad (3.19c) \\
    r_t &= r_{t-1} + \alpha (\mu - r_{t-1}) \Delta t + \sigma \sqrt{\Delta t} \sqrt{|r_{t-1}|} Z_t \quad (3.19d)
\end{align*}
\]

where \(X^+ = \max(X,0), X \in \mathbb{R}\). However, only (3.19c) guarantees to generate nonnegative values in all scenarios. Given \(r_{t-1}\), Equations (3.19a), (3.19b) and (3.19d) give identical
(but possibly negative) values when \( r_{t-1} \geq 0 \). When \( r_{t-1} < 0 \), both (3.19a) and (3.19b) drive \( r_t \) back towards the long-term mean with (3.18a) giving a larger value of \( r_t \), and the sign of \( r_t \) depends on the value of \( \alpha \) and \( \mu \). The value of \( r_t \) given by (3.19d), on the other hand, can be either larger or smaller than \( r_{t-1} \) depending on \( Z_t \). Table 3.1 shows some examples of how the above four schemes work when \( r_{t-1} \) is close to zero or negative.

Table 3.1: Illustrations of four schemes to deal with negative rates under a discretized CIR model. Values of \( r_t \) given \( r_{t-1} \) and \( Z_t \)

<table>
<thead>
<tr>
<th>( r_{t-1} )</th>
<th>( r_{t-1} = 10^{-4} )</th>
<th>( r_{t-1} = -10^{-4} )</th>
<th>( r_{t-1} = -10^{-4} )</th>
<th>( r_{t-1} = -10^{-2} )</th>
<th>( r_{t-1} = -10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_t )</td>
<td>( Z_t = -2.2288 )</td>
<td>( Z_t = -2.2288 )</td>
<td>( Z_t = 2.7557 )</td>
<td>( Z_t = -2.2288 )</td>
<td>( Z_t = 2.7557 )</td>
</tr>
<tr>
<td>(3.19a)</td>
<td>0.0002246</td>
<td>0.000903</td>
<td>0.000903</td>
<td>-0.008750</td>
<td>-0.008750</td>
</tr>
<tr>
<td>(3.19b)</td>
<td>0.0002246</td>
<td>0.000900</td>
<td>0.000900</td>
<td>-0.009000</td>
<td>-0.009000</td>
</tr>
<tr>
<td>(3.19c)</td>
<td>0.0002246</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>(3.19d)</td>
<td>0.0002246</td>
<td>-0.0002441</td>
<td>0.005037</td>
<td>-0.042182</td>
<td>0.032586</td>
</tr>
</tbody>
</table>

Note: \( \alpha = 0.1, \mu = 0.04, \sigma = 0.3 \) and \( \Delta_t = 0.25 \).

The chosen values of \( Z_t \), -2.2288 and 2.7557, are the extreme values we deemed in a set of 10,000 random draws from the standard normal distribution.

None of the above methods are needed in this project because the volatility parameter of the interest rate model is set at a small enough value that no negative values of \( r_t \) are encountered.

### 3.4 Numerical Results

#### 3.4.1 Vasicek Model

In the previous section, we have discussed the distribution of profits and losses when hedging a single-premium GMDB under a constant risk-free rate framework. In this section, the interest rate is stochastic and assumed to follow a Vasicek model. As for the parameters of the Vasicek model, we choose the maximum likelihood estimates from Zhou (2007) which are based on historical interest rates. The long-term mean of the interest rate is assumed to be about 0.04-0.05 and the long-term standard deviation is about 0.013-0.015. Our chosen parameters are as follows:

- \( r_0 = 0.02 \) (starting interest rate);
- \( \alpha = 0.1 \) (mean-reverting rate);
- \( \mu = 0.04 \) (long-term mean risk-free rate);
- \( \frac{\sigma}{\sqrt{2\alpha}} = 0.013 \) (long-term standard deviation).
Figures 3.2 to 3.5 show the simulated risk-free rate at the end of years 1, 5 and 10, as well as projected values of \( r_t \) over a 10-year period. We can conclude that the standard deviation of the risk-free rate increases with time. The standard deviation of \( r_t \) approaches \( \sigma \sqrt{2\alpha} \) after some time and then becomes stable.

Now that we have projected the interest rate for 10 years at a weekly frequency, it is of interest to study the distribution of the hedging errors and the profits under a delta hedging strategy. The approach used in Section 2.2 is repeated. We simulate 10,000 scenarios, but this time using stochastic interest rates, meaning that every week the hedging portfolio is updated using the current interest rate and the hedging error is tracked. The distribution of the profit \( \Pi \) and hedging error paths are illustrated in Figures 3.6 and 3.7. The results
show a larger standard deviation of hedging errors compared to the case of constant risk-free rate. One should expect this as another source of randomness is added to the model.

![Figure 3.6: Distribution of profits and losses for the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.](image1)

![Figure 3.7: Paths of hedging error for the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.](image2)

### 3.4.2 Sensitivity Test, Vasicek Model

The parameter $\alpha$ represents how fast the risk-free rate reverts back to the long-term mean. The next step is to explore the impact of the mean-reverting speed on the distribution of the hedging errors while the long-term volatility remains unchanged. Figures 3.8 to 3.13 show the simulated paths of $r_t$, the distributions of profit and loss, and hedging errors for two different levels of $\alpha$. As one would expect, when $\alpha$ is larger the risk-free rate reverts back to the long-term mean more quickly and the volatility also approaches the long-term volatility more quickly. For example, $r_t$ is still not close to its stationary state at maturity when $\alpha$ is 0.01. But when $\alpha$ is 1, it is already approaching its stationary state after a few years into the contract. In fact, with $\alpha = 0.01$, it would take more than 100 years for the volatility of $r_t$ to get close to its long-term volatility. In our case, as the long-term volatility is fixed, when $\alpha$ is larger, the local volatility $\sigma$ is also larger. It can be observed from Figures 3.8 and 3.9 that given time $t$, the distribution of $r_t$ has a larger standard deviation when $\alpha = 1$, leading to a larger standard deviation of the distribution of profit.
Figure 3.8: Simulated paths of $r_t$ for the Vasicek model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2}\alpha} = 0.013$.

Figure 3.9: Simulated paths of $r_t$ for the Vasicek model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2}\alpha} = 0.013$.

Figure 3.10: Distribution of profits and losses for the Vasicek model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2}\alpha} = 0.013$.

Figure 3.11: Distribution of profits and losses for the Vasicek model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2}\alpha} = 0.013$. 
3.4.3 CIR Model

One advantage of the CIR model is that \( r_t \) remains positive provided \( 2\alpha \mu \geq \sigma^2 \). Therefore, it is essential to choose reasonable parameters. In order for the results to be comparable with those of the Vasicek model, the parameters should be chosen so that the long-term means and long-term standard deviations under both models are the same. Recall that the long-term mean and long-term standard deviation are \( \mu_v \) and \( \frac{\sigma_v}{\sqrt{2\alpha_v}} \) under the Vasicek model; and \( \mu_c \) and \( \frac{\sigma_c}{\sqrt{2\alpha_c}} \sqrt{\mu_c} \) under the CIR model, respectively. Theoretically we should have

\[
\mu_c = \mu_v \tag{3.20a}
\]
\[
\frac{\sigma_v}{\sqrt{2\alpha_c}} = \frac{\sigma_c}{\sqrt{2\alpha_c}} \sqrt{\mu_c}. \tag{3.20b}
\]

Therefore, the parameters for the CIR model are:

- \( r_0 = 0.02 \) (starting interest rate);
- \( \alpha = 0.1 \) (mean-reverting rate);
- \( \mu = 0.04 \) (long-term mean risk-free rate);
- \( \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013 \) (long-term standard deviation).

Figures 3.14 to 3.17 illustrate simulated distributions of \( r_1, r_5 \) and \( r_{10} \) as well as simulated paths of \( r_t \) over a 10-year period. Compared with the interest rate simulated using the Vasicek model, one can conclude that the distribution of \( r_t \) at a given time \( t \) is less symmetric and has smaller standard deviation under the CIR model. For example, the simulated values of \( r_1 \) range from 0.005 to 0.04 under the CIR model and from -0.005 to 0.045 under the Vasicek model. This is due to the fact that the local volatility varies with \( r_t \) under the
CIR model. The local volatility becomes very close to zero when \( r_t \) is very small. In fact, the distribution of \( r_t \) conditional on \( r_0 \) is no longer a normal distribution. Meanwhile, since the starting value of the interest rate, \( r_0 \), is only 0.02 and \( r_t \) is expected to still be relatively small at maturity when \( \alpha \) equals 0.1, the local volatility remains fairly small throughout the term of the contract. Therefore, the standard deviations of the distribution of \( r_t \) are smaller than those under the Vasicek model.

\[
\text{Figure 3.14: Distribution of } r_1 \text{ for the CIR model using } \alpha = 0.1, r_0 = 0.02, \mu = 0.04 \text{ and } \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013.
\]

\[
\text{Figure 3.15: Distribution of } r_5 \text{ for the CIR model using } \alpha = 0.1, r_0 = 0.02, \mu = 0.04, \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013.
\]

\[
\text{Figure 3.16: Distribution of } r_{10} \text{ for the CIR model using } \alpha = 0.1, r_0 = 0.02, \mu = 0.04 \text{ and } \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013.
\]

\[
\text{Figure 3.17: Simulated paths of } r_t \text{ for the CIR model using } \alpha = 0.1, r_0 = 0.02, \mu = 0.04 \text{ and } \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013.
\]

Although under continuous time \( r_t \) remains positive under the CIR model, provided \( 2\alpha \mu \geq \sigma^2 \), it is possible for negative values of \( r_t \) to occur when the CIR model is discretized with a time step of one week corresponding to the frequency at which we choose to rebalance.
the portfolio. Note that we did not observe any negative value of \( r_t \) with our chosen long-term volatility of \( \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} \), which is equal to 0.013. But it is of interest to understand how often \( r_t \) becomes negative even with the condition \( 2\alpha \mu \geq \sigma^2 \) being satisfied. To this end, different long-term volatilities and mean-reverting speeds are chosen, and a floor that is very close to zero is set. Then by studying the frequency at which \( r_t \) hits the floor we can study the frequency at which \( r_t \) approaches zero.

First, the number of scenarios that hit the floor at least once is identified and the results are reported in Table 3.2. Then, among those scenarios we find the one with the highest frequency and report the results in Table 3.3. The floor is set to be 0.0001. The long-term volatility stays fixed for each column while \( \sigma \) varies with respect to \( \alpha \). In all cases the condition \( 2\alpha \mu \geq \sigma^2 \) is satisfied. From Table 3.2, we can conclude that given the same long-term volatility, the larger the mean-reverting speed \( \alpha \), the higher the probability that \( r_t \) becomes negative. Given the same mean-reverting speed, the larger the long-term volatility, the higher the chance of \( r_t \) becoming negative. The same approach is applied to the Vasicek model to study the frequency of generating negative values of \( r_t \) (see Appendix A for the results).

Table 3.2: Percentage of scenarios where \( r_t \) hits the floor under the CIR model

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} ) = 0.03</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0126</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.2526</td>
<td>0.0004</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: \( \mu \) is fixed and equals 0.04, \( \sigma \) varies with respect to \( \alpha \).

Table 3.3: Largest number of times (weeks) \( r_t \) is below the floor in one scenario under the CIR model

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} ) = 0.03</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: \( \mu \) is fixed and equals 0.04, \( \sigma \) varies with respect to \( \alpha \).

When \( \alpha \) is 0.1, \( r_t \) has not reached its stationary state at maturity under either the Vasicek or the CIR model. But with a small \( r_0 \) and an interest-rate dependent local volatility, the distribution of \( r_t \) given time \( t \) has a smaller standard deviation under the CIR model even though the long-term volatilities are the same under both models. Therefore, one can expect the standard deviation of the hedging error and the profit to be smaller than those under the Vasicek model. The results are illustrated in Figures 3.18 and 3.19.
3.4.4 Sensitivity Test, CIR Model

Similar sensitivity tests are conducted to explore the impact of the mean-reverting speed on the distribution of the profit and loss under the CIR model. Figures 3.20 to 3.25 show the simulated paths of $r_t$ and the hedging errors, as well as the distribution of profit and loss for different values of $\alpha$. Similar to the situation when $\alpha = 0.1$, we observe lower volatilities of $r_t$ than under the Vasicek model when $\alpha = 0.01$. This is due to the slow mean-reverting speed, the small value of $r_0$ and the interest rate dependent volatility feature of the CIR model. When $\alpha = 1$, we observed approximately the same long-term standard deviation ($r_{t0}$ ranges from 0 to 0.12) as in the Vasicek model (recall that $r_{t0}$ ranges from -0.02 to 0.1 in the Vasicek model). This is due to the fact that $r_t$ reverts to its long-term mean faster when $\alpha = 1$. 

Figure 3.18: Distribution of profits and losses for the CIR model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}}\sqrt{\mu} = 0.013$. 

Figure 3.19: Paths of hedging error for the CIR model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}}\sqrt{\mu} = 0.013$. 

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Figure 3.20: Simulated paths of $r_t$ for the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$, $\frac{\sigma}{\sqrt{2\pi}}\sqrt{\mu} = 0.013$.

Figure 3.21: Simulated paths of $r_t$ for the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}}\sqrt{\mu} = 0.013$.

Figure 3.22: Distribution of profits and losses for the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}}\sqrt{\mu} = 0.013$.

Figure 3.23: Distribution of profits and losses for the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}}\sqrt{\mu} = 0.013$. 
Figure 3.24: Paths of hedging error for the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}} \sqrt{\mu} = 0.013$.

Figure 3.25: Paths of hedging error for the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\pi}} \sqrt{\mu} = 0.013$. 
Chapter 4

Delta Hedging Under Annual Management Fee Rate

4.1 Annual Management Fee Rate

In the previous chapters, we did simulations under the assumption that the insurer charges an upfront management fee. However, in reality, life insurance companies usually charge management fees at a flat rate every month. In this chapter, a management fee is assumed to be charged each time the hedging portfolio is rebalanced. Let us denote the weekly fee rate as \( m \) (the hedging portfolio is rebalanced on a weekly basis), then the fund value at time \( t \), immediately before the next weekly fee is deducted, can be expressed as:

\[
F_t = (1 - m)^{52t} F_0 \frac{S_t}{S_0}, \quad t = 0, \frac{1}{52}, \frac{2}{52}, \ldots, 1, \frac{1}{52}, \ldots, 10. \tag{4.1}
\]

Note that the management fee at each time is \( mF_t \).

Let \( G \) denote the guarantee value. The payoff for the GMMB then becomes \((G - F_T)^+\) and the price of the GMMB using the Black and Scholes framework can be written as

\[
P_0 = e^{-rT}E[(G - F_T)^+] \\
= e^{-rT}E[(G - S_T(1 - m)^{52T})^+] \\
= Ge^{-rT}\Phi(-d_2) - S_0(1 - m)^{52T}\Phi(-d_1) \tag{4.2}
\]
where
\[ d_1 = \frac{\ln(S_0(1-m)^{52T}/G) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \], 
\[ d_2 = d_1 - \sigma\sqrt{T} \].

This pricing formula assumes that all policyholders survive to the maturity date of the contract.

The option price for a GMMB replicating portfolio that allows for exits, e.g. death or lapse, is multiplied by the survival probability. But here we assume that the mortality risk and the lapse risk are hedged by diversification. It also implies that the lapse risk is assumed to be independent of the guaranteed liability, which is generally not true.

To construct a hedging portfolio at time \( t \), we need \( \Delta P(t) \) shares of stock and \( B_t \) invested in the money-market account. Under the Black and Scholes framework, \( \Delta P(t) \) and \( B_t \) can be expressed as:
\[ \Delta P(t) = -S_t(1-m)^{52t}\Phi(-d_1), \]
\[ B_t = Ge^{-r(T-t)}\Phi(-d_2), \quad t = 0, \frac{1}{52}, \frac{2}{52}, \ldots, 1, \frac{1}{52}, \ldots, 10. \]

where
\[ d_1 = \frac{\ln(S_t(1-m)^{52t}/G) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \]
\[ d_2 = d_1 - \sigma\sqrt{T-t}. \]

### 4.2 Numerical Results

#### 4.2.1 Vasicek Model

In our numerical example, an annual management fee rate of 1\% is assumed to be charged weekly on a pro rated basis. We are interested in comparing the hedging effectiveness and the profit under two different types of fee charges: an upfront fee charge and annual fee charges. When a 10\% management fee is charged to a fund that has initial value of 100, 90 dollars are invested in the underlying stock index and then the 10 dollars are added to the payoff at maturity to calculate the profits or losses for the insurer. Now, instead of charging 10 dollars of management fees at issue, \( \frac{1}{52} \)\% of the fund value is charged as a management fee at each step of hedging. The total charge is no longer 10 dollars. In fact, when the equity market performs well, the actual fee charge increases and so the insurer earns more profits. When the equity market crashes, the actual fee charge decreases drastically and so
the insurer experiences bigger losses. Figure 4.1 shows simulated paths of management fees charged at each weekly rebalancing point during the whole contract period.

![Figure 4.1: Simulated weekly management fees under 1% annual fee rate.](image)

Figures 4.2 to 4.13 compare the distributions of hedging error as well as profits and losses for different values of $\alpha$ under two types of fee charges. The figures on the left hand side show the results with an annual management fee rate and figures on the right hand side show the results with upfront management fee. One can conclude that the profit under an annual fee rate charge has a larger standard deviation than under an upfront fee charge. This is consistent with what we would expect since under an annual rate charge, when there is a bull market, the management charge increases and thus the profits are larger. At the same time a bear market would be worse for the insurer as less management fee is charged and larger losses emerge. However, when $\alpha = 0.1$, a bear market does not suggest a worse performance. This is possible because the total charge depends on the path of the stock price, which further depends on $\sigma$ and $\mu$. For example, it could happen that the equity market performs well during the first few years and then drops below the guarantee value near the maturity date. In this case, although the payoff is negative at maturity, the total charges are larger and the insurer still can make a profit.
Figure 4.2: Distribution of cumulative hedging error at maturity with management expense ratio charge under the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.

Figure 4.3: Distribution of cumulative hedging error at maturity with upfront fee charge under the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.

Figure 4.4: Distribution of profits and losses with management expense ratio charge under the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.

Figure 4.5: Distribution of profits and losses with upfront fee charge under the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$. 

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Figure 4.6: Distribution of cumulative hedging error at maturity with management expense ratio charge under the Vasicek model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\sigma \sqrt{2\alpha} = 0.013$.

Figure 4.7: Distribution of cumulative hedging error at maturity with upfront fee charge under the Vasicek model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\sigma \sqrt{2\alpha} = 0.013$.

Figure 4.8: Distribution of profits and losses with management expense ratio charge under the Vasicek model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\sigma \sqrt{2\alpha} = 0.013$.

Figure 4.9: Distribution of profits and losses with upfront fee charge under the Vasicek model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\sigma \sqrt{2\alpha} = 0.013$. 

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4.2.2 CIR Model

The same annual management fee rate (1%) is applied to the CIR model to compare the hedging effectiveness as well as profits and losses with both types of fee charges. The results shown in Figures 4.14 to 4.25 are quite similar to those of the Vasicek model. As the actual fees charged based on annual rate are higher during bull markets and lower during bear markets when compared to upfront charges, one should expect higher profits as well as larger losses when using the annual management fee rate.
Figure 4.14: Distribution of cumulative hedging error at maturity with management expense ratio charge under the CIR model using $\alpha = 0.1,$ $r_0 = 0.02,$ $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$.

Figure 4.15: Distribution of cumulative hedging error at maturity with upfront fee charge under the CIR model using $\alpha = 0.1,$ $r_0 = 0.02,$ $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$.

Figure 4.16: Distribution of profits and losses with management expense ratio charge under the CIR model using $\alpha = 0.1,$ $r_0 = 0.02,$ $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$.

Figure 4.17: Distribution of profits and losses with upfront fee charge under the CIR model using $\alpha = 0.1,$ $r_0 = 0.02,$ $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$. 

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Figure 4.18: Distribution of cumulative hedging error at maturity with management expense ratio charge under the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}}\sqrt{\mu} = 0.013$.

Figure 4.19: Distribution of cumulative hedging error at maturity with upfront fee charge under the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}}\sqrt{\mu} = 0.013$.

Figure 4.20: Distribution of profits and losses with management expense ratio charge under the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}}\sqrt{\mu} = 0.013$.

Figure 4.21: Distribution of profits and losses with upfront fee charge under the CIR model using $\alpha = 0.01$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}}\sqrt{\mu} = 0.013$. 
Figure 4.22: Distribution of cumulative hedging error at maturity with management expense ratio charge under the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$.

Figure 4.23: Distribution of cumulative hedging error at maturity with upfront fee charge under the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$.

Figure 4.24: Distribution of profits and losses with management expense ratio charge under the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$.

Figure 4.25: Distribution of profits and losses with upfront fee charge under the CIR model using $\alpha = 1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} \sqrt{\mu} = 0.013$. 
Chapter 5

Regime-Switching Lognormal Model

5.1 Introduction

So far, we have studied the distribution of profits when a delta hedging strategy is used and when the stock price is assumed to follow a geometric Brownian motion. However, using the GBM to model stock prices has some drawbacks, one of which is that it assumes the volatility to be constant over time. Historically, it has been observed that the volatility of returns in the equity market is higher in bear markets and lower in bull markets. A regime-Switching lognormal model can help overcome this problem.

The regime-switching lognormal (RSLN) model was introduced by Bollen (1998) who used the model to value American options. RSLN models assume that the process is, at any given time, in one of \( K \) regimes where each regime is driven by a GBM with its own mean and variance. The RSLN model has the advantage of capturing the variable volatility while keeping the simplicity of the independent lognormal model.

Under a RSLN model, we assume the stock return is generated in one of the \( K \) regimes. Let \( \rho_t \) denote the regime in which the stock return is generated from time \( t \) to \( t + 1 \) and let \( Y_t \) be the log-return of the stock index in this period, that is \( Y_t = \ln(S_{t+1}/S_t) \). Then

\[
Y_t | \rho_t \sim N(\mu_{\rho_t}, \sigma^2_{\rho_t})
\] (5.1)

where \( \mu_i \) and \( \sigma^2_i \) are the mean and variance of the \( i \)th regime.

The simplest regime-switching model is the two-regime RSLN model, which is found to be good enough to fit the stock index data (see, for example, Hardy, 2003, p.31). Let \( p_{i,j} \)
denote the probability that the regime of the process switches from Regime $i$ to Regime $j$ at the end of a given period. Then

$$ p_{i,j} = \Pr[\rho_{t+1} = j | \rho_t = i], \quad i = 1, 2, \quad j = 1, 2, \quad \forall t. \quad (5.2) $$

This gives the transition matrix $P$:

$$ P = \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix} \quad (5.3) $$

with $p_{i,1} + p_{i,2} = 1$. There are therefore six parameters that need to be estimated in total, i.e.,

$$ \Theta = \{\mu_1, \mu_2, \sigma_1, \sigma_2, p_{1,2}, p_{2,1} \}. \quad (5.4) $$

5.2 Maximum Likelihood Estimation for RSLN-2 Model

The RSLN model can be estimated using the maximum likelihood method (see, for example, Hardy, 2003, and Hamilton, 2010). Consider the following regime switching model, which is simply a convenient way to rewrite Equation (5.1):

$$ y_t | \rho_t = \mu_{\rho_t} + \sigma_{\rho_t} \epsilon_t \quad (5.5) $$

where $\epsilon_t$ is an independent, identically distributed standard normal random variable, $N(0, 1)$. The log-likelihood function for this model, given that we know the regime path, is as follows:

$$ \ln L = \sum_{t=1}^{T} \ln \left( \frac{1}{\sqrt{2\pi\sigma_{\rho_t}^2}} \exp \left( -\frac{y_t - \mu_{\rho_t}}{2\sigma_{\rho_t}^2} \right) \right) \quad (5.6) $$

where $\rho_t = 1, 2$.

If all of the regimes $\rho_t$ are known, then the maximum likelihood estimation only requires maximizing the equation above, as a function of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$. However, under a RSLN-2 model, the states are not observed. Therefore, we have to rewrite the log-likelihood using conditional distributions. Let the available information at time $t-1$ be denoted as $\psi_t$, the log-likelihood function can be written as:

$$ \ln L = \sum_{t=1}^{T} \ln \sum_{j=1}^{2} f(y_t | \rho_t = j, \psi_{t-1}, \Theta) \Pr(\rho_t = j | \psi_{t-1}) \quad (5.7) $$
which is in fact a weighted average of the log-likelihood functions for each regime and the weights are the regimes’ conditional probabilities. One is not able to use the equation directly as these probabilities are unknown. Hamilton and Susmel (1994) suggest that Hamilton’s filter can be applied to calculate the probabilities recursively based on the new arrival information.

To start the iterative estimation algorithm, one needs to set starting probabilities, which are generally chosen to be the stationary (unconditional) probabilities. Let us assume \( \pi = (\pi_1, \pi_2) \) is the unconditional distribution for the process. Under stationarity conditions, the distribution \( \pi = (\pi_1, \pi_2) \) is time-invariant, which means:

\[
\pi P = \pi, \tag{5.8}
\]

that is

\[
\pi_1 p_{1,1} + \pi_2 p_{2,1} = \pi_1 \tag{5.9}
\]

and

\[
\pi_1 p_{1,2} + \pi_2 p_{2,2} = \pi_2. \tag{5.10}
\]

Since

\[
p_{1,1} + p_{1,2} = 1, \tag{5.11}
\]

we have that

\[
\pi_1 = \frac{p_{2,1}}{p_{1,2} + p_{2,1}}, \tag{5.12a}
\]

\[
\pi_2 = 1 - \pi_1 = \frac{p_{1,2}}{p_{1,2} + p_{2,1}}. \tag{5.12b}
\]

Now, since we have the unconditional probabilities, the first step is to set the starting probabilities of each regime (or state) as their unconditional probabilities:

\[
\Pr(\rho_0 = 1|\psi_0) = \frac{p_{1,2}}{p_{1,2} + p_{2,1}}, \tag{5.13a}
\]

\[
\Pr(\rho_0 = 2|\psi_0) = \frac{p_{2,1}}{p_{1,2} + p_{2,1}}. \tag{5.13b}
\]
The second step is to calculate the probabilities of each state given the information up to time $t - 1$, which is:

$$
\Pr(\rho_t = j | \psi_{t-1}) = \sum_{i=1}^{2} p_{ji}(\Pr(\rho_{t-1} = i | \psi_{t-1})),
$$

(5.14)

where $p_{ji}$ is the transition probability to state $i$ at time $t$ given that state is $j$ at time $t - 1$.

The third step is to update the probability of each state adding the information at time $t$. To achieve this, one should calculate the likelihood function in each state. Given $\rho_t$, the likelihood function of $y_t$ has no dependence on earlier information, and

$$
f(y_t | \rho_t = j, \psi_{t-1}) = \frac{1}{\sigma_j} \phi(\frac{(y_t - \mu_j)}{\sigma_j}).
$$

(5.15)

Then, we can use the following formula to update the probability of each state using new information

$$
\Pr(\rho_t = j | \psi_t) = \frac{f(y_t | \rho_t = j, \psi_{t-1})\Pr(\rho_t = j | \psi_{t-1})}{\sum_{j=1}^{2} f(y_t | \rho_t = j, \psi_{t-1})\Pr(\rho_t = j | \psi_{t-1})}.
$$

(5.16)

For example, we can start the recursion with

$$
f(y_1, \rho_1 = j | \Theta) = \frac{p_{1.2}}{p_{1.2} + p_{2.1}} \frac{1}{\sigma_1} \phi(\frac{(y_1 - \mu_1)}{\sigma_1}),
$$

(5.17a)

$$
f(y_2, \rho_2 = j | \Theta) = \frac{p_{1.2}}{p_{1.2} + p_{2.1}} \frac{1}{\sigma_2} \phi(\frac{(y_2 - \mu_2)}{\sigma_2}),
$$

(5.17b)

and update the probabilities of each state to be used in the next recursion using

$$
\Pr(\rho_1 = j | y_1, \Theta) = \frac{f(y_1 | \rho_1 = j, \Theta)\Pr(\rho_1 = j | \Theta)}{\sum_{j=1}^{2} f(y_1 | \rho_1 = j, \Theta)\Pr(\rho_t = j | \Theta)}.
$$

(5.18)

We then repeat steps two and three until $t = T$.

The maximum likelihood estimation consists of finding the set of parameters that maximizes the equation

$$
\ln L = \sum_{t=1}^{T} \ln \sum_{j=1}^{2} f(y_t | \rho_t = j, \Theta)\Pr(\rho_t = j | \psi_t).
$$

(5.19)

### 5.3 Estimation Result

Using the MLE method above, we fit the RSLN-2 model to historical values of the S&P 500 index (see Figure 5.1). Figure 5.2 shows the monthly returns of S&P 500 index. The
estimation results are shown in Table 5.1. Monthly data from 1956-2001 are used in the estimation and our results are generally consistent with the results found in Hardy (2003).

Table 5.1: MLE parameters for RSLN-2 model with estimated standard error.

<table>
<thead>
<tr>
<th>S&amp;P 500</th>
<th>( \hat{\mu}_1 = 0.0097(0.002) )</th>
<th>( \hat{\sigma}_1^2 = 0.001202(0.0001) )</th>
<th>( \hat{p}_{1,2} = 0.04 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\mu}_2 = -0.0162(0.014) )</td>
<td>( \hat{\sigma}_2^2 = 0.004551(0.0013) )</td>
<td>( \hat{p}_{2,1} = 0.26 )</td>
<td></td>
</tr>
</tbody>
</table>

Further analysis is conducted to investigate the MLE parameters using different periods of the S&P 500 index. The historical data of the S&P index indicates two peaks during the past 15 years. The equity market crashed in 2001 and 2008 and started recovering in 2009. We are interested in investigating how the MLE’s are influenced by the time span we choose. The additional MLE results are shown in Table 5.2.

Figure 5.1: Historical S&P 500 index from 1955 to 2016 (source: Yahoo Finance)
Figure 5.2: Historical S&P 500 monthly return from 1955 to 2016
Here, Regime 1 represents the bull market while Regime 2 stands for the bear market. The results show that during different time periods of the same length, the returns during bull markets and bear markets can be very different. For example, the average return during bull markets for 1955-1970 is 0.0163, which is higher than that for 1970-1985 (only 0.0051). The average return during bear markets is around -0.009, but there are exceptions: the negative return is small (only -0.0029) for 1955-1970, but the negative return during bear markets for 1985-2000 is significantly larger than any of the other periods. It is largely due to the crash of Black Monday in 1987. In addition, the transition probabilities are very different for different periods. For instance, the transition probability from a bull market to a bear market for 1955-1970 is 0.13, which is larger than the transition probability from a bear market to a bull market. This means the stock index spent more time in bear markets than in bull markets, which in fact contradicts the general understanding of the equity market in real world. Therefore, we choose to use all the data from 1955 to 2016 to estimate the parameters of the RSLN model.
Table 5.3: Comparisons of MLE parameters for RSLN-2 model with estimated standard error.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\mu}_1 )</td>
<td>0.0103(0.002)</td>
<td>0.0025(0.0003)</td>
</tr>
<tr>
<td>( \hat{\sigma}_1^2 )</td>
<td>0.0000992(0.0002)</td>
<td>0.000215(0.0000)</td>
</tr>
<tr>
<td>( \hat{\nu}_1 )</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>( \hat{\mu}_2 )</td>
<td>-0.0085(0.010)</td>
<td>-0.0024(0.0013)</td>
</tr>
<tr>
<td>( \hat{\sigma}_2^2 )</td>
<td>0.003701(0.0009)</td>
<td>0.001100(0.0001)</td>
</tr>
<tr>
<td>( \hat{\nu}_2 )</td>
<td>0.15</td>
<td>0.06</td>
</tr>
</tbody>
</table>

In addition, it is of interest to investigate whether the frequency of the data has an impact on our MLE results. Table 5.3 shows the estimation results using weekly data from 1955 to 2016. The table shows that the frequency of the data does not have a significant impact on the estimated model. For example, the estimated average return based on weekly data is 0.0025, which, when multiplied by 4, is approximately the same as the average return of 0.0103 based on monthly data. The time period, however, does matter. In other words, using weekly data instead of monthly data within the same time period does not affect the estimated model significantly. In order to get more reliable estimates of the model, one should try to include as many years of data as possible.

5.4 Stock Price Simulation

With the historical stock index fitted to a RSLN-2 model, we are able to use the MLE parameters to simulate stock paths for 10 years. Figure 5.3 shows simulated stock prices using the MLE parameters estimated from returns over 1955-2016. From the figure, we can tell that the maximum value of the stock price is no more than 700, which is much less than the maximum value of 1600 obtained under the GBM model.
Figure 5.3: Simulated stock paths for the RSLN-2 estimated using 1955-2016 S&P 500 data.

Table 5.4: Comparison of stock price under two models.

<table>
<thead>
<tr>
<th>Model</th>
<th>E($S_{10}$)</th>
<th>SD($S_{10}$)</th>
<th>Max ($S_{10}$)</th>
<th>Min ($S_{10}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSLN-2</td>
<td>111.31</td>
<td>56.47</td>
<td>520.42</td>
<td>14.19</td>
</tr>
<tr>
<td>GBM</td>
<td>181.76</td>
<td>126.40</td>
<td>1601.00</td>
<td>10.07</td>
</tr>
</tbody>
</table>

SD: Standard deviation

Recall that the expected annual return used to simulate stock paths under the GBM model is 0.06, which is approximately the same as the estimated annual return under the RSLN-2 model (0.0612) based on weekly data. The monthly volatility parameter used to simulate stock prices under the GBM is 0.0577 (which is obtained from $\sqrt{0.02}$). The MLE volatility parameter is 0.0314 during bull markets and 0.0608 during bear markets, and the average is about 0.03875. However, the simulated stock paths are different from those results, especially on the upside of stock price. This is reasonable as the volatility in bull markets is less than 0.0577, which is much smaller than that under GBM. Therefore, the growth of stock prices is limited in bull markets under RSLN-2 model, leading to lower maximum value. As for bear markets, although the volatility is higher than 0.05, since the stock price is very low, the high volatility has little influence.
5.5 Delta Hedging Under RSLN-2 Model

Using new stock paths generated under the RSLN-2 model, it might be interesting to see how the new stock prices influence the distribution of profits and losses for the insurer. Suppose the fund manager uses the Black and Scholes model to construct the hedging portfolio with the same parameters as in Chapter 2, while the stock index actually follows a RSLN-2 model. By assuming that the interest rate follows a Vasicek model with the same parameters as in the previous chapters and that the insurer charges an annual fee, we obtain the distribution of profits and hedging errors with different mean-reverting parameters as shown in Figures 5.4 to 5.9.

![Figure 5.4: Distribution of cumulative hedging error at maturity for the RSLN-2 model using \( \hat{\mu}_1 = 0.0103, \hat{\mu}_2 = -0.0085, \hat{\sigma}_1^2 = 0.000992, \hat{\sigma}_2^2 = 0.003701, \hat{p}_{1,2} = 0.05 \) and \( \hat{p}_{2,1} = 0.15 \); and the Vasicek model using \( \alpha = 0.01, r_0 = 0.02, \mu = 0.04 \) and \( \frac{\sigma}{\sqrt{2\alpha}} = 0.013 \).](image1)

![Figure 5.5: Distribution of profits and losses for the RSLN-2 model using \( \hat{\mu}_1 = 0.0103, \hat{\mu}_2 = -0.0085, \hat{\sigma}_1^2 = 0.000992, \hat{\sigma}_2^2 = 0.003701, \hat{p}_{1,2} = 0.05 \) and \( \hat{p}_{2,1} = 0.15 \); and the Vasicek model using \( \alpha = 0.01, r_0 = 0.02, \mu = 0.04 \) and \( \frac{\sigma}{\sqrt{2\alpha}} = 0.013 \).](image2)
Compared with the distributions of hedging error and profit using simulated stock paths under the GBM model, it is obvious that the distribution of hedging errors using stock prices from a RSLN-2 model is less skewed and has larger standard deviations. Besides, one reason this occurs is that the insurer is less likely to earn huge profits under the RSLN-2 model since the growth in stock price is much more restricted than under the GBM model, leading to less opportunities for profit. In addition, the stock price is more volatile in bear markets,
resulting in larger losses when the equity market performs very poorly. Generally speaking, if the stock index follows a RSLN-2 process but the fund manager uses the Black and Scholes model to hedge the guarantee, the hedging errors can become very large. By charging an annual fee, it is still possible for the insurer to earn some profits; however, the profits will not be as high as under the GBM model. In addition, the insurer is more likely to experience larger losses.

Noting that the hedging is not effective when a delta hedging strategy with weekly rebalancing is implemented, it is of interest to see whether increasing the trading frequency will reduce the hedging errors. Figures 5.10 and 5.11 show the distributions of hedging errors as well as profits and losses when implementing a delta hedging strategy with daily rebalancing.

![Figure 5.10: Distribution of cumulative hedging error at maturity with daily rebalancing for the RSLN-2 model using $\hat{\mu}_1 = 0.0103$, $\hat{\mu}_2 = -0.0085$, $\hat{\sigma}_1^2 = 0.000992$, $\hat{\sigma}_2^2 = 0.003701$, $\hat{p}_{1,2} = 0.05$ and $\hat{p}_{2,1} = 0.15$; and the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.](image1)

![Figure 5.11: Distribution of profits and losses with daily rebalancing for the RSLN-2 model using $\hat{\mu}_1 = 0.0103$, $\hat{\mu}_2 = -0.0085$, $\hat{\sigma}_1^2 = 0.000992$, $\hat{\sigma}_2^2 = 0.003701$, $\hat{p}_{1,2} = 0.05$ and $\hat{p}_{2,1} = 0.15$; and the Vasicek model using $\alpha = 0.1$, $r_0 = 0.02$, $\mu = 0.04$ and $\frac{\sigma}{\sqrt{2\alpha}} = 0.013$.](image2)

Compared with the hedging errors and profits under weekly rebalancing (Figures 5.6 and 5.7), it seems that the hedging effectiveness is not improved at all when rebalancing daily. In fact, when the Black-Scholes assumptions are violated, delta hedging alone is no longer sufficient to reduce the risk. There are more associated risks such as volatility and interest rate.
Chapter 6

Conclusion

This project discusses delta hedging of a segregated fund guarantee, a product which has grown drastically in market value during the past few years. The Guaranteed Minimum Maturity Benefit protects the investor when the equity market performs poorly which can cause significant losses to the insurer. Under the Black and Scholes framework, where stock price is assumed to follow a geometric Brownian motion and both the risk-free rate and the volatility are assumed to be constant, a delta hedging strategy successfully reduces the risk when the hedging portfolio is rebalanced at a reasonable frequency, for example, weekly.

However, as the term of a segregated fund is long, assuming a constant risk-free rate is not reasonable. Two popular one factor short-rate models are used to model the interest rate during the whole contract period. It is not surprising to see that the hedging errors become larger under stochastic interest rates as the Black and Scholes assumptions are violated. But as long as the mean-reverting speed of the interest rate is not very high, we can still get good hedging effectiveness using the Black and Scholes formula. In addition, the type of management fee also has an impact on hedging effectiveness. Charging an annual fee as a percentage of assets results in larger profits or losses for the insurer than charging an upfront fee because the actual amount charged by the insurer under the annual rate is larger when the equity market performs well and smaller during bear markets.

The geometric Brownian motion is easy to implement, but is not consistent with empirical evidence. A more realistic model for the stock index is the regime-switching lognormal model. It is interesting to see that hedging errors can be very large and huge losses for the insurer may occur when the actual stock price follows a RSLN-2 model and the fund manager delta hedges the guarantee by assuming a geometric Brownian motion. It is also shown that increasing the hedging frequency cannot reduce the losses.
6.1 Further Work

In summary, if the Black and Scholes assumptions were never violated, stock prices would be the only risk that one would have to worry about in order to hedge the guarantee. However, this is not the case in real markets where there are other risks involved. Any variable that is incorporated into the price of the GMMB but does not stay fixed over time is a source of risk. Therefore it is essential to consider hedging other risks. For instance, instead of using delta hedging, one can also hedge the gamma of the GMMB, which is a second-order measure that shows the sensitivity of the GMMB delta to the underlying stock prices. There are also other hedging strategies such as Vega hedging which hedges the volatility risk and rho hedging which hedges the risk from changes in interest rates.

Generally speaking, delta hedging is an effective hedging method. But it can generate large hedging errors in the presence of jump risk. A large movement size per se is not an issue as it can still be hedged by rebalancing the hedging portfolio at a higher frequency. It is the instantaneous volatility risk that is problematic. One may consider local risk minimization hedging under a Merton’s jump diffusion model (Coleman et al., 2007), which has been demonstrated to be less sensitive to stock price changes than the alternative delta hedging.
Bibliography


Appendix

Table A.1: Vasicek model - Percentage of scenarios generating negative value of $r(t)$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{\sigma}{\sqrt{2\alpha}} = 0.03$</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0938</td>
<td>0.0117</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4915</td>
<td>0.2246</td>
<td>0.0031</td>
</tr>
<tr>
<td>1</td>
<td>0.9298</td>
<td>0.6259</td>
<td>0.0063</td>
</tr>
</tbody>
</table>

Table A.2: Vasicek model - Largest number of times (weeks) $r(t)$ is negative in one scenario

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{\sigma}{\sqrt{2\alpha}} = 0.03$</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>388</td>
<td>311</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>509</td>
<td>472</td>
<td>195</td>
</tr>
<tr>
<td>1</td>
<td>343</td>
<td>207</td>
<td>31</td>
</tr>
</tbody>
</table>