# Analysis of Universal Life Insurance Cash Flows with Stochastic Asset Models

by

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B.Sc., Nankai University, 2013

Project Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Statistics and Actuarial Science
Faculty of Science


**SIMON FRASER UNIVERSITY**
**Summer 2016**

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# Abstract

Universal life insurance is a flexible product which provides the policyholder with life insurance protection as well as savings build-up. The performance of the policy is hard to be evaluated accurately with deterministic asset models, especially when the fund is placed in accounts that track the performance of equities. This project aims to investigate factors that affect the savings (account value) and insurance coverage (death benefit) under a stochastic framework. Time series models are built to capture the complex dynamics of returns from two commonly offered investment options, T-bills and S&P 500 index, with and without interdependence assumption. Cash flows of account value, cost of insurance, and death benefit are projected for sample policies with common product features under multiple investment strategies. The comparison reveals the impact of asset models and fund allocation on the projected cash flows.

**Keywords:** Universal life insurance; Stochastic asset modelling; Cash flow projection; Time series model; Account value

# Dedication

*To my beloved parents, for their love, support, encouragement, and sacrifice.*

# Acknowledgements

Foremost, I would like to express my profoundest gratitude to my senior supervisor Dr. Yi Lu. Her professional guidance, unlimited encouragement, immense knowledge, and unreserved help were key motivations throughout my graduate study and research. She let me find my own way whilst showing me alternative paths I might have missed. When working on this project with Dr. Lu, she taught me the vital skill of disciplined critical thinking and proper attitudes towards every challenge I face. Without her kind and patient instruction, this project would never come into fruition.

Meanwhile, I would like to extend my gratitude to the rest of my committee: Dr. Cary Tsai and Ms. Barbara Sanders, who have read my project carefully, commented on countless revisions, and provided constructive suggestions. I wish to thank Dr. Tim Swartz for being the chair of the examining committee. And I hugely appreciate the generous help and advice from Dr. Gary Parker, who squeezed time from his busy schedule to examine my work and kindly kept his office door open for any question.

I am also indebted to my graduate fellows. They shared my happiness, offered listening ears when I was depressed, and helped me to believe in myself. We together created the memory that I will cherish forever.

Last but not the least, thanks go to my parents for everything. They made me into who I am.

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# Chapter 1

# Introduction

## 1.1 Background and Motivation

Prior to the introduction of universal life insurance products in the early 1980s, people bought traditional permanent insurance products such as whole life insurance and endowment for lifelong insurance protection. The contract terms for such traditional products are "fixed and guaranteed", which means:

- premiums must be paid when due for a specific number of years or to a specific age;
- death benefits are guaranteed at issue and do not change without underwriting unless the policy has riders modifying the death benefit, or is participating and the dividend is used to purchase additional insurance.

A distinguishing feature of permanent products is that they develop significant cash values which can be thought of as accumulated premiums less mortality costs and expenses. With the label of "fixed and guaranteed", the cash values are guaranteed as well.

Since the 1980s, policyholders had an alternative: universal life (UL) insurance. It was initially created to take advantage of the unprecedented increase in short- and mid-term interest rates in the 1980s and has grown to be the dominant form of permanent life insurance coverage in North America according to the statistics from the Life Insurance Marketing and Research Association (LIMRA) (Wark, 2003).

In general, UL insurance combines long-term insurance protection and tax-advantaged investment in a customizable policy. Neither are premiums predetermined, nor does the continuation of the policy depend on payment of premiums. Policyholders decide the amount, timing, and allocation of premiums based on their individual circumstances but within limits specified in the contract. Due to the flexibility, the cash values of universal life products are not determined up front. Instead, they are valued on an ongoing basis by accumulating

an account value for a particular UL policy[1]. The premiums are added to the policyholder's account value, and expense and mortality charges are deducted from the account value. The account value is deposited into investment account(s) per the policyholder's choice and earns interest. Varying in investment account(s) offered, two types of universal life insurance have been developed. The policyholder of universal life (UL) insurance is not involved in the investment decision. The entire fund is placed in a general account where the credited rate is determined by the insurance company based on the performance of the company's general asset portfolio. In contrast, variable universal life (VUL) insurance provides a variety of investment accounts at different risk levels. Besides the general account, options range from conservative bond-based accounts to risky equity-linked and mutual fund accounts. Policyholders could manage their money based on their risk profiles, preferred asset types, and investment styles. Depending on the death benefit option chosen by the policyholder, the death benefit is either the face amount of the policy (Type A) or the face amount plus the account value (Type B), subject to corridor requirements.

Since the death benefit that the beneficiary receives is linked to the policy's account value, it is important from an insurance planner's standpoint to have a sense of possible future cash flows, or account values to be more specific. However, increasingly complex product design has complicated the projection of cash flows and account values. If the fund is heavily invested in risky accounts such as equity-linked accounts, the rates of return fluctuate in a manner that traditional deterministic models can not capture. "Loadings" on assumptions regarding credited rates may be insufficient as preparation for bad extremes in reality.

To provide a more accurate illustration of future outcomes, stochastic modelling is utilized in this project. The first objective of this project is to establish stochastic models for returns of a simplified asset portfolio made up of a safe investment (T-bills) and a risky asset (an equity index) based on historical data. By generating a large scenario set, a thorough evaluation of future account values and resulting death benefits is obtained. The goal is to figure out how premium allocation strategies and asset models impact policy lapses, the growth of account values, and the determination of the death benefit under a stochastic framework. The influence of stochastic analysis is also studied through the comparison with deterministic projection results.

## 1.2 Literature Review

There are not many academic papers on the actuarial evaluation of cash flows under universal life insurance. The existing studies are from different perspectives, summarized as

[1]The official definition of the cash value for a universal life policy is the amount the policyholder receives when the policy lapses and equals the account value less the surrender charge. Thus the cash value and account value are, to some extent, equivalent except for the gap incurred by surrender charges.

follows.

A number of studies focus on the evaluation of the investment performance of (V)UL policies. D'Arcy and Lee (1987) compared the after-tax cash values of a VUL policy with combinations of term insurance and different investment alternatives assuming that the investor (policyholder) purchased an equivalent amount of insurance coverage. Using industry average data to determine parameter values, they showed how to choose the optimal investment vehicle depending on the policyholder's expected holding period. Some studies are conducted with respect to the marginal rate of return and internal rate of return of universal life insurance. Cherin and Hutchins (1987) assessed the internal rates of return of 60 universal life policies and compared them with the corresponding advertised interest rates. They concluded that the investor would be better off buying term insurance in the open market and investing the remaining premiums in alternative investment instruments rather than paying for a universal life policy, and that expense and mortality charges accounted for the discrepancy. D'Arcy and Lee (1989) calculated the optimal contribution to a VUL policy based on the marginal rate of return and the internal rate of return when the policyholder with a given level of capital wanted the insurance coverage within or outside variable universal life insurance while maximizing the after-tax rate of return.

The determinants of UL insurance cash values have also been studied. Chung and Skipper (1987) found a positive correlation between the current interest rate and projected cash values for durations of ten years or longer in a univariate framework. Carson (1996) conducted a multivariate analysis and demonstrated that the effects of expense, mortality, and surrender charges outweigh the effects of interest rates for one-year and five-year periods.

## 1.3 Outline

This project is organized as follows. Chapter 2 introduces the products to be analysed and presents the methodology for calculating basic cash flows: premium, expense, cost of insurance, account value, and death benefit. In Chapter 3, we establish univariate and multivariate time series models to describe the dynamics of the returns of the asset portfolio consisting of US 3-month Treasury Bills and the S&P 500 index. Chapter 4 explains how the simulation analysis is conducted and presents numerical results of all projected cash flows. Discussions are also provided regarding the impact of stochastic investment returns from policyholder's point of view. Chapter 5 concludes the project.

# Chapter 2

# Mechanism of Universal Life Insurance

All elements of cash flows for typical universal life insurance policies are variable. Premiums are not predetermined; within fairly wide limits the policyholder has flexibility in premium payment pattern. Upon deducting expenses, premiums are deposited into a notional account (or accounts if the policyholder seeks diversification by building an investment portfolio) and are broken down into insurance and savings components according to the purposes they serve. The insurance portion, referred to as the cost of insurance (COI), pays for the coverage provided by the insurer and is proportional to the amount of death benefit which is not supported by policyholder's own fund. This amount measures the risk borne by the insurance company and thus is called the net amount at risk (NAAR). The savings portion is the remainder of the premiums and is invested to build the account value. The account value is the balance in the policyholder's account (or the total balance in all accounts), and it makes up the rest of policyholder's death benefit (DB) besides the NAAR promised by the insurer. In this sense, mortality risks are shared by the insurance company and the policyholder.

Two death benefit options are usually available. Adopting the naming convention in Atkinson and Dallas (2000), a Type A universal life insurance policy has a constant death benefit, whereas a Type B policy maintains a level NAAR allowing the death benefit to increase with the account value. The ratio of the death benefit to the account value is subject to corridor factors to ensure that the savings component does not overwhelm the insurance coverage. Unlike traditional products, policyholders could use their savings to pay for expenses and COI charges. Therefore, continuation of a universal life policy is tied to having enough fund in the account to cover regular deductions rather than payment of pre-scheduled premiums.

Universal life insurance products credit interest, subject to a guaranteed minimum rate

if applicable. The insurance company decides the interest rate to be credited based on earnings on the company's general asset portfolio or a financial index. The policyholder does not choose investments. One variation of the universal life insurance, called variable universal life (VUL) insurance, offers both hands-off and hands-on approaches to investment. The policyholder could allocate premiums to a wide selection of investment accounts. One basic account, called the savings account in this project, is tied to the performance of the company's general asset portfolio. The others credit interest based on returns of reference assets including bonds, market indices, and actively managed mutual funds. No minimum interest rate is guaranteed except for the amount allocated to the savings account. The insurance company bears the investment risk for the fund in the savings account, while all investment risks are transferred to policyholders for funds in the other accounts.

In this project, we consider VUL insurance as our standard product to allow for more flexibility from the policyholder's perspective. The rest of this chapter firstly presents the calculation of cash flows including account value, COI, expense, and death benefit, secondly discusses how those cash flows change under corridor requirements, and finally introduces how premiums are determined.

## 2.1 Accumulation of Account Value

We begin with an introduction of notation. Let

$\omega$ = limiting age of mortality table,

$x$ = issue age,

$q_x$ = probability of death between ages $x$ and $x+1$,

$e$ = expense charge rate,

$\kappa$ = substandard multiplier rating,

$FA$ = face amount, superscripts A and B are used afterwards to distinguish between Type A and Type B products,

$v_{t-1}$ = death benefit discount factor applied in policy year $t$,

$c_x$ = corridor factor at attained age $x$,

$i_{t-1}$ = credited rate in policy year $t$, and

$i^G$ = guaranteed minimum credited rate.

Following Dickson et al. (2013), we consider a universal life insurance policy issued to an insured aged $x$. The cash flows are updated annually, and no partial withdrawal is allowed. At the beginning of $t$th policy year, $t = 1, 2, \ldots, \omega - x$, the policyholder pays the premium

denoted by $P_{t-1}$. Unless otherwise noted, subscript regarding time is as of the exact payment time. After the premium payment, an expense charge $E_{t-1}$ is deducted. Assume that the expense charge is $\$e$ per $1 premium. Then

$$E_{t-1} = eP_{t-1}. \tag{2.1}$$

In return for the death benefit $DB_t$ payable upon the death of the insured in year $t$, a cost of insurance (COI) charge denoted by $COI_{t-1}$ is deducted from the policyholder's fund. The COI is the product of the COI rate and the net amount at risk at the end of policy year $t$ discounted back to the start of the year; that is

$$COI_{t-1} = v_{t-1}\, q^r_{x+t-1}\, NAAR_t, \tag{2.2}$$

where $q^r_{x+t-1}$ is the COI rate applied to the policyholder attaining age $x+t-1$ in policy year $t$, and $NAAR_t$ is the net amount at risk at the end of policy year $t$. It can be interpreted as the single premium for a one-year term insurance with sum insured equal to the $NAAR_t$.

The COI rate varies for the insured in different risk classes. Taking risk level into account, the COI rate $q^r_{x+t-1}$ is adjusted from the standard mortality rate (or unrated COI rate) according to following rating system:

$$q^r_{x+t-1} = \kappa q_{x+t-1}, \tag{2.3}$$

where $\kappa$ is the substandard multiplicative rating that distinguishes lives at different risk levels and $q_{x+t-1}$ is the standard rate. Note that $\kappa$ is 1 for the insured with no rating (i.e., standard life), otherwise $\kappa > 1$.

The net amount at risk (NAAR) is the liability of the insurer in exchange for COI charges. It is the portion of the death benefit not supported by policyholder's fund; that is

$$NAAR_t = DB_t - AV_t, \tag{2.4}$$

where $DB_t$ is the death benefit if death occurs in policy year $t$, and $AV_t$ is the policyholder's account value at the end of policy year $t$. In this sense, the NAAR measures the risk the insurer bears in dollar amount. The choice of death benefit option influences how the NAAR is calculated. For a Type A policy, the death benefit in policy year $t$, $DB_t$, is level throughout the policy period and is specified in the contract as

$$DB_t = FA^A. \tag{2.5}$$

Provided (2.4), the $NAAR_t$ decreases as $AV_t$ increases. For a Type B policy, the face amount is the amount of coverage provided by the insurer; in other words,

$$NAAR_t = FA^B. \tag{2.6}$$

Given (2.4), the death benefit becomes

$$DB_t = AV_t + NAAR_t = AV_t + FA^B.$$

Then the legacy received by the beneficiary of a Type B policy varies with the account value.

The account value accumulates by $i_{t-1}$ after premiums flowing in and charges deducted. In policy year $t$, the account value at the year end, $AV_t$, is obtained by

$$AV_t = (AV_{t-1} + P_{t-1} - E_{t-1} - COI_{t-1})(1 + i_{t-1}), \quad t = 1, 2, \ldots, \omega - x.$$

With (2.1), the equation becomes

$$AV_t = \Big[AV_{t-1} + (1-e)P_{t-1} - COI_{t-1}\Big](1 + i_{t-1}). \tag{2.7}$$

Provided (2.2) and (2.3), we further have

$$AV_t = \Big[AV_{t-1} + (1-e)P_{t-1} - \kappa v_{t-1} q_{x+t-1} NAAR_t\Big](1 + i_{t-1}). \tag{2.8}$$

Theoretically, starting from $AV_0 = 0$, the account value in each policy year can be calculated recursively by (2.8).

## 2.2 Corridor Requirement and Recalculated COI

According to the US Internal Revenue Code (1986), the relation of the death benefit and the account value must pass either the guideline level premium test (GLPT) or the cash value accumulation test to ensure that a life insurance policy is used predominantly for insurance purposes and is eligible for tax exemption. In this project, a GLPT is conducted so that the death benefit must equal or exceed a certain multiple of the account value, which as a result constrains the net amount at risk. Based on Dickson et al. (2013), the corridor death benefit is defined as

$$DB_t^c = c_{x+t-1}\ AV_t, \tag{2.9}$$

where $c_{x+t-1}$ is the corridor factor applied to an insured at age $x + t - 1$. As one ages, the corridor factor declines to 1. The corridor death benefit is the minimum death benefit

required, given the attained age of the insured and the account value, to ensure that policy is compliant. If the current death benefit $DB_t$ is no less than corridor death benefit $DB_t^c$, the policy passes the corridor test. Otherwise, the death benefit needs to be raised to the corridor death benefit, and the COI charge requires recalculation.

Consider a Type A product first. If the corridor test is satisfied, the net amount at risk, $NAAR_t$, is determined by (2.4). Given (2.5),

$$NAAR_t^f = FA^A - AV_t,$$

where the superscript $f$ indicates that all calculations are based on the face amount. With (2.7), the net amount at risk becomes

$$NAAR_t^f = FA^A - \Big[AV_{t-1} + (1-e)P_{t-1} - COI_{t-1}^f\Big](1+i_{t-1}). \tag{2.10}$$

Combining (2.10), (2.2), and (2.3), we could solve for the COI charge

$$COI_{t-1}^f = \frac{\kappa q_{x+t-1} v_{t-1}\Big[FA^A - (AV_{t-1} + (1-e)P_{t-1})(1+i_{t-1})\Big]}{1 - \kappa q_{x+t-1} v_{t-1}(1+i_{t-1})}.$$

If the current death benefit does not reach the floor set by the corridor requirement, corridor factors play a role in determining the death benefit and the COI charge. With (2.9) and (2.4), the net amount at risk is given by

$$NAAR_t^c = (c_{x+t-1} - 1)AV_t, \tag{2.11}$$

where the superscript $c$ indicates application of corridor factors. Similarly from (2.11), (2.2), and (2.3), we could get

$$COI_{t-1}^c = \frac{\kappa q_{x+t-1} v_{t-1}(c_{x+t-1} - 1)(1+i_{t-1})(AV_{t-1} + (1-e)P_{t-1})}{1 + \kappa q_{x+t-1} v_{t-1}(c_{x+t-1} - 1)(1+i_{t-1})}. \tag{2.12}$$

By comparing $COI_{t-1}^f$ and $COI_{t-1}^c$, we could judge whether the corridor factor comes into effect. A larger NAAR contributes to a larger COI charge. Thus if $COI_{t-1}^c = \max(COI_{t-1}^f, COI_{t-1}^c)$, the corridor requirement leads to an increase of the COI charge to $COI_{t-1}^c$. Otherwise, the regular charge $COI_{t-1}$ (i.e., $COI_{t-1}^f$) is imposed on the account. Conclusively,

$$COI_{t-1} = \max(COI_{t-1}^f,\ COI_{t-1}^c), \tag{2.13}$$

and

$$DB_t = \max(FA^A,\ c_{x+t-1}\ AV_t).$$

On the contrary, for a Type B policy, the net amount at risk always equals the face amount. With (2.2), (2.3), and (2.6), the COI based on the face amount becomes

$$COI_{t-1}^{f} = \kappa q_{x+t-1}\ v_{t-1}\ FA^{B}.$$

Both (2.12) and (2.13) still hold, while the death benefit becomes

$$DB_t = \max(FA^{B} + AV_t,\ c_{x+t-1}\ AV_t).$$

Then we could decide the COI and death benefit in any policy year.

## 2.3 Guaranteed Maturity Premium

The premium payment pattern $\{P_t\}$ highly depends on the policyholder's behaviour because of the flexibility declared by the contract. Policyholders act in their best interest by nature in aspects of premium payment amount and frequency. For example, if the credited rate is low at the moment, the policyholder may suspend payments and use the account value to cover all the deductions. When the credited rate bounces back, the policyholder may continue building up the account value. Other factors such as external interest rates, the level of surrender charges, and the tax deferral function of universal life products, may also impact the premium payment pattern. Consequently, it is hard to predict this pattern with confidence. However, if the investment account is designed with a guaranteed credited rate, a planned premium, referred to as the guaranteed maturity premium (GMP), could shed light on how much one needs to pay to keep the policy in force throughout the policy period.

According to Lombardi (2006), the guaranteed maturity premium, denoted as $P^G$, is the annual level gross premium that provides for an endowment for the face amount payable at the latest permissible maturity date under the contract. It is calculated as of the issue date with guarantees as to expense charges, cost of insurance charges, and credited interest rates. In most cases, actual expense charges equal the guaranteed maximum expense charges, whereas actual mortality charges are often less than the guaranteed maximum. For simplicity, we assume that the guaranteed and actual charges on both mortality and expense are the same. The guaranteed minimum credited rate, $i^G$, is usually locked in after the issue date, and hence it is assumed to be deterministic.

The mechanism of the account value accumulation on a non-guaranteed basis, as discussed in Section 2.1, also applies to the fund accumulation on a guaranteed basis. Hence, by letting all assumptions in (2.8) be the guaranteed levels, the guaranteed account value at

the end of policy year 1 for both product types is obtained as

$$AV_1 = \begin{cases} (AV_0 + (1-e)P^G - \kappa v_{t-1} q_{x+1}\ (FA^A - AV_1))(1+i^G), & \text{Type A,} \\ (AV_0 + (1-e)P^G - \kappa v_{t-1} q_{x+1}\ FA^B)(1+i^G), & \text{Type B,} \end{cases} \tag{2.14}$$

where $i^G$ and $P^G$ are the guaranteed credited rate and the guaranteed maturity premium, respectively, and $AV_0 = 0$. From (2.14), we see that $AV_1$ is a function of $P^G$. Therefore, we could write

$$AV_1 \triangleq \begin{cases} f_1^A(P^G), & \text{Type A,} \\ f_1^B(P^G), & \text{Type B.} \end{cases}$$

Likewise in policy year 2, with (2.8), (2.4), (2.5), and (2.6), we have

$$\begin{aligned} AV_2 &= \begin{cases} (AV_1 + (1-e)P^G - \kappa v_{t-1} q_{x+1}\ (FA^A - AV_2))(1+i^G), & \text{Type A,} \\ (AV_1 + (1-e)P^G - \kappa v_{t-1} q_{x+1}\ FA^B)(1+i^G), & \text{Type B;} \end{cases} \\ &\triangleq \begin{cases} f_2^A(f_1(P^G), P^G), & \text{Type A,} \\ f_2^B(f_1(P^G), P^G), & \text{Type B;} \end{cases} \\ &\triangleq \begin{cases} F_2^A(P^G), & \text{Type A,} \\ F_2^B(P^G), & \text{Type B.} \end{cases} \end{aligned}$$

In any policy year $i$, $i = 1, 2, \ldots, \omega - x$, we have

$$AV_i = \begin{cases} F_i^A(P^G), & \text{Type A,} \\ F_i^B(P^G), & \text{Type B,} \end{cases}$$

implying that the account value depends on $P^G$. Then letting $AV_{\omega-x} = 0$, we can solve for the guaranteed maturity premium $P^G$. Numerical procedures are needed in this process.

# Chapter 3

# Stochastic Processes of Assets

The prevailing universal life insurance products in the marketplace usually provide multiple investment options which track the performances of a broad array of investment instruments such as Treasury Bills (T-bill), bonds, equity indices, and mutual funds. Policyholders could tailor their premium allocation strategy to reflect their objectives and risk tolerance. In this project, the investment strategy from the policyholder's standpoint is simplified to allocation between two investment accounts, a savings account and an equity account, where credited rates are linked to 3-month T-bill rates and the yields on the S&P 500 index, respectively. Depending on whether the asset model captures the interdependence between these two assets, two modelling schemes are outlined in the remainder of this chapter.

## 3.1 Univariate Model

An intuitive way to model the combination of the two accounts is to assume that the returns of two assets fluctuate independently and that each is modelled by a separate process.

### 3.1.1 Model for US T-bills

Figure 3.1 displays the monthly observations of 3-month US T-bill rates (nominal, per annum) from Jan. 1982 to Aug. 2015[1] and the rolling one-year volatilities. Descriptive statistics are calculated and presented in Table 3.1. The overall trend of nominal rates is downward except for a few bounces in 1982, 1986, 1994, 1999, and 2004. In 2008, the rate hit the bottom and has stayed low since then. Rises of the rolling standard deviation coincided with the dramatic ups and downs of the nominal rates. We employ an autoregressive moving average (ARMA) model to capture the path dependence of the data.

[1] Data are obtained from US Federal Reserve System.

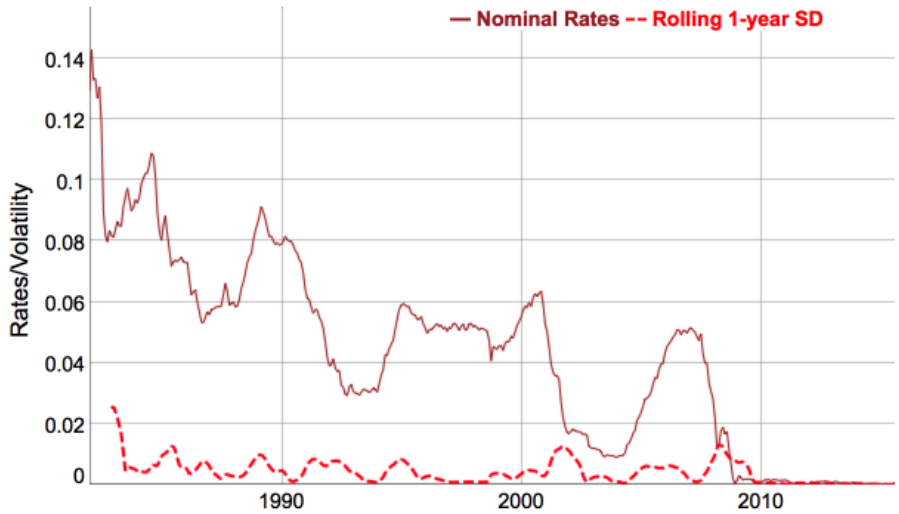



Figure 3.1: Historical nominal annual rates and annual volatility, US 3-month T-bill

| Mean | 0.0425 | Quantiles | |
|---|---|---|---|
| Standard deviation | 0.0314 | 5% | 0.0003 |
| Min | 0.0001 | 25% | 0.0121 |
| Max | 0.1428 | Median | 0.0472 |
| Skewness | 0.3412 | 75% | 0.0597 |
| Excess kurtosis | -0.3860 | 95% | 0.0924 |

Table 3.1: Summary statistics of nominal annual rates, US 3-month T-bill

To ensure a consistent time unit between observations and interest rates, monthly rates of return are modelled. Let $\{r_t\}$ denote the sequence of monthly T-bill rates. Log transformation is conducted on $r_t$ to prevent negative rates. Then an ARMA$(p, q)$ model (Box et al. , 1994) on $ln(r_t)$ is of the form

$$ln(r_t) - \mu = \sum_{i=1}^{p} \phi_i \Big( ln(r_{t-i}) - \mu \Big) + \varepsilon_t - \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}. \tag{3.1}$$

In (3.1), $\mu$ is the long-term mean of the process, $\phi_i$'s and $\theta_j$'s are the coefficients of the autoregressive (AR) and the moving average (MA) terms, respectively, and $\{\varepsilon_t\}$ are innovations that incorporate the fluctuation not explained by the past values with $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$. Using the back-shift operator $B$, (3.1) is rewritten as

$$(1 - \phi_1 B - \cdots - \phi_p B^p)\Big( ln(r_t) - \mu \Big) = (1 - \theta_1 B - \cdots - \theta_p B^q)\varepsilon_t.$$

The autocorrelation function (ACF) and partial autocorrelation function (PACF) offer effective tools for order identification. Alternatively, one could compare information criteria such as the Akaike information criterion (AIC) (Akaike, 1973), and select $p$ and $q$ that give a minimum AIC value.

Tsay (2005) has shown that an ARMA$(p, q)$ model as (3.1) can be inverted into an infinite-order autoregressive model; that is, there are coefficients $\pi_j$, $j = 1, 2, \ldots$, such that

$$ln(r_t) - \mu = \pi_1\Big(ln(r_{t-1}) - \mu\Big) + \pi_2\Big(ln(r_{t-2}) - \mu\Big) + \pi_3\Big(ln(r_{t-3}) - \mu\Big) + \cdots + \varepsilon_t$$

if the MA characteristic roots, which satisfy

$$1 - \theta_1 B - \cdots - \theta_p B^q = 0,$$

exceed 1 in modulus. Then the residuals are defined as

$$\begin{aligned} e_t &= ln(r_t) - ln(\hat{r}_t) \\ &= ln(r_t) - \hat{\mu} - \hat{\pi}_1\Big(ln(r_{t-1}) - \hat{\mu}\Big) - \hat{\pi}_2\Big(ln(r_{t-2}) - \hat{\mu}\Big) - \hat{\pi}_3\Big(ln(r_{t-3}) - \hat{\mu}\Big) - \ldots, \end{aligned}$$

where $\hat{\pi}_i$'s [2] and $\hat{\mu}$ are parameters estimated from the data.

A sound ARMA model should satisfy that residuals act like white noise, that parameter estimates are significant, and that the process is stationary. These characteristics are examined as Table 3.2 suggests.

| **Properties** | **Tests** |
|---|---|
| Significance of coefficient | – two-sided t-test,<br>– p-value less than the significance level indicates the coefficient is significantly different from zero. |
| Stationarity | – AR characteristic roots exceed 1 in absolute value (Cryer and Chan, 2008). |
| Normality of residuals | – Shapiro-Wilk normality test, W statistic (Shapiro and Wilk, 1965),<br>– p-value exceeding significance level indicates normal residuals. |
| Independence of residuals | – Q statistic (Ljung and Box, 1978), asymptotically chi-square distributed,<br>– p-value exceeding the significance level indicates no serial correlation in residuals. |

Table 3.2: Model diagnostics, ARMA$(p, q)$

[2] The $\pi$'s are not estimated directly but rather implicitly as functions of $\phi$'s and $\theta$'s.

Before fitting to the model, historical nominal annual rates are converted to monthly rates through dividing by 12 and then transformed by the logarithm function. The ACF and PACF of log monthly rates are shown in Figure 3.2. The decay of the autocorrelation function is quite slow, and strong correlation still exists after a 25-month lag. The PACF is cut off at lag 3. This suggests that AR(1), AR(2), and ARMA(2,1) are worth consideration.

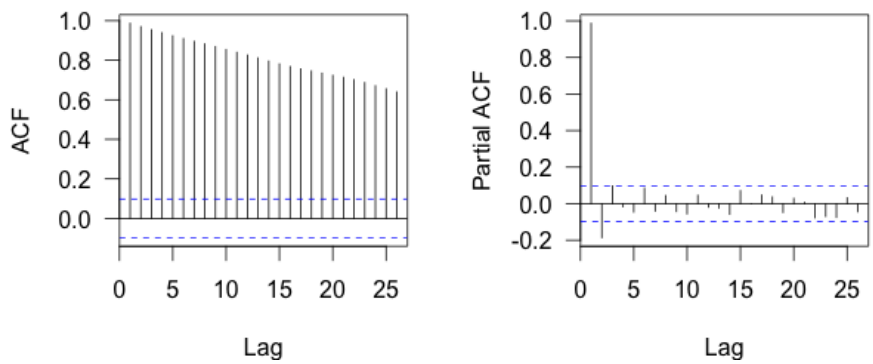



Figure 3.2: Autocorrelation and partial autocorrelation functions, log monthly T-bill rates

Model estimation and diagnosis are done by R, and results are summarized in Table 3.3. All models are stationary and incorporate significant AR (and MA) terms. AR(1) fails the Ljung-Box test as the p-value of the Q statistic is relatively small. Among the rest, ARMA(2,1) gives a smaller AIC value; therefore it is chosen as the functional form of $ln(r_t)$.

| **Model** | **Parameters** (p-values) | **AIC** | **Characteristic roots** | **Q statistics** (p-values) | **W statistics** (p-values) |
|---|---|---|---|---|---|
| AR(1) | $\phi_1$: 0.9943 (0.0000)<br>$\mu$:−6.7530 (9.3699e-7)<br>$\sigma_\varepsilon^2$: 0.0514 | −44.71 | 1.0057 | 10.5482<br>(0.0011) | 0.6287<br>($< 2.2e{-}16$) |
| AR(2) | $\phi_1$: 1.1636 (0.0000)<br>$\phi_2$:−0.1709 (1.6278e-4)<br>$\mu$:−6.7456 (1.1652e-8)<br>$\sigma_\varepsilon^2$: 0.0499 | −53.89 | 1.0089<br>5.7992 | 0.4684<br>(0.4937) | 0.6287<br>($< 2.2e{-}16$) |
| ARMA(2,1) | $\phi_1$: 0.7333 (1.1754e-08)<br>$\phi_2$: 0.2564 (1.3875e-02)<br>$\theta_1$: 0.4893 (7.1767e-06)<br>$\mu$:−6.6157 (3.5960e-08)<br>$\sigma_\varepsilon^2$: 0.0487 | −62.32 | 1.0082<br>−3.8678[3] | 0.1433<br>(0.705) | 0.6352<br>($< 2.2e{-}16$) |

Table 3.3: Estimation results, AR(1), AR(2), and ARMA(2,1)

[3]MA characteristic roots are not listed because they do not determine stationarity.

According to Table 3.3, the ARMA(2,1) model for log monthly T-bill rates is formulated as

$$ln(r_t) + 6.6157 = 0.7333\Big[ln(r_{t-1}) + 6.6157\Big] + 0.2564\Big[ln(r_{t-2}) + 6.6157\Big] + \varepsilon_t - 0.4893\varepsilon_{t-1}, \tag{3.2}$$

where $\varepsilon_t \sim N(0, 0.0487)$. Note that the normality of residual is not validated, which implies a limitation of the above model.

### 3.1.2 Model for S&P 500 Index

Another category of popular investment accounts are equity-linked accounts. Here the Standard & Poor's 500 (S&P 500) Index is chosen as an example for analysis. Figure 3.3 shows monthly rates of return on the S&P 500 index with rolling 1-year volatility estimates from Feb. 1950 to Sep. 2015[4]. These returns are approximated by the growth of closing prices on the last day of successive months. Table 3.4 shows descriptive statistics of the historical returns.

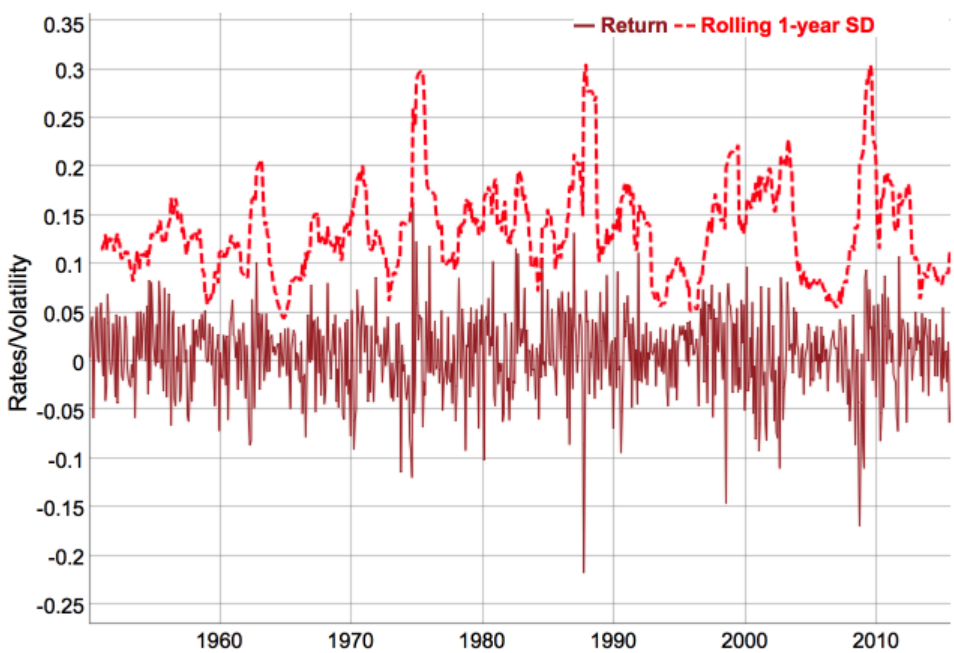



Figure 3.3: Monthly rates of return and annual volatilities, S&P 500

[4]Data are from Yahoo Finance.

| Mean | 0.0069 | Quantiles | |
|---|---|---|---|
| Standard deviation | 0.0415 | 5% | −0.0621 |
| Min | −0.2176 | 25% | −0.0176 |
| Max | 0.1630 | Median | 0.0091 |
| Skewness | −0.4221 | 75% | 0.0350 |
| Kurtosis | 1.7102 | 95% | 0.0709 |

Table 3.4: Summary statistics of monthly rate of return, S&P 500

This time period covers several bull and bear markets with a series of sharp stock market declines such as those seen in Oct. 1987, Aug. 1998, and Sep. 2008. A notable feature exposed by Figure 3.3 is that the volatilities vary over time. As noted by Mandelbrot (1963), "large changes (in price) tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes". This phenomenon, commonly referred to as volatility clustering, breaks the fundamental assumption of ARMA models that the innovations are independent and identically distributed. To improve the model so that it allows for volatility bunching, a generalized autoregressive conditional heteroscedasticity (GARCH) process (Bollerslev, 1986) is added to the classical ARMA model.

Let $\delta_t$ be the log return (i.e., force of interest) of the S&P 500 index on a monthly basis at time $t$. Unlike the pure ARMA$(p, q)$ model where errors are i.i.d. standard normal random variables, the model with a GARCH process treats the variances of the innovations as a time series which evolves over time. Specifically, the process governing the development of $\delta_t$ is written as

$$\delta_t = a_t + \varepsilon_t, \tag{3.3a}$$

$$a_t = \phi_0 + \sum_{i=1}^{p} \phi_i a_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}, \tag{3.3b}$$

$$\varepsilon_t = \sigma_t z_t, \tag{3.3c}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2. \tag{3.3d}$$

The change of $\delta_t$ is driven by two forces: $a_t$, the conditional mean of $\delta_t$ given the past information up to time $t-1$, and the innovation $\varepsilon_t$. The conditional mean $a_t$ follows an ARMA$(p, q)$ model with a constant term $\phi_0$ as described by (3.3b). Meanwhile, given the standardized error $z_t$ such that $z_t \overset{iid}{\sim} N(0, 1)$, the innovation $\varepsilon_t$ is constrained by the conditional variance of $\delta_t$, $\sigma_t^2$. With the past information up to time $t-1$, $\sigma_t^2$ follows a GARCH$(m, s)$ model given by (3.3d) where $\alpha_0 > 0$, $\alpha_i \geq 0$ for $i = 1, 2, \ldots, m$, $\beta_j \geq 0$ for $j = 1, 2, \ldots, s$, and $\sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i) < 1$.

In terms of identifying the mean process (3.3a), the aforementioned technique for ARMA models still applies. The GARCH orders $m$ and $s$, as Cryer and Chan (2008) stated, are determined by fitting the squared residuals of the main process, $\varepsilon_t^2$, to an ARMA model and identifying the orders $p^{res}$ and $q^{res}$. The GARCH($m$,$s$) model for $\{\delta_t\}$ implies an ARMA($\max(m, s), s$) model for the squared residuals. Hence, $s$ and $m$ are obtained by

$$s = q^{res},$$
$$0 \leq m \leq p^{res}.$$

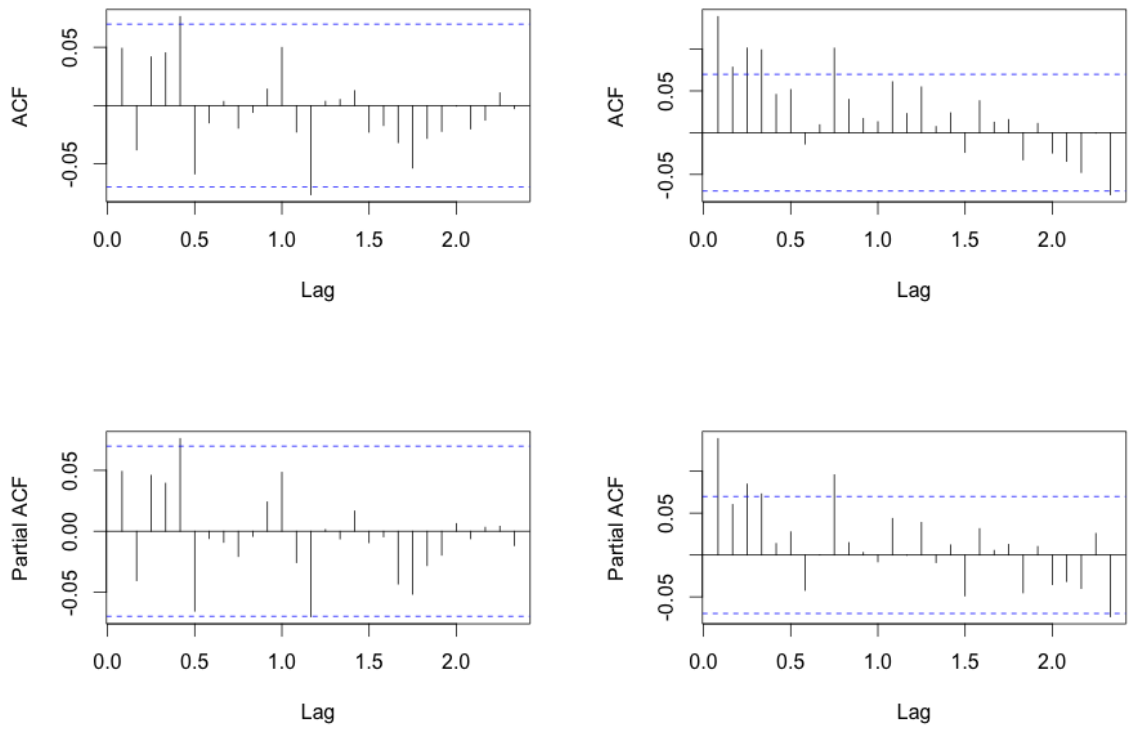
To ensure the model assumptions are supported, a series of tests are conducted as Table 3.5 shows.

| **Properties** | **Tests** |
|---|---|
| Significance of coefficient | – two-sided t-test; p-value less than the significance level indicates the coefficient is significantly non-zero. |
| Stationarity | – $\sum_{i=1}^{max(m,s)}(\alpha_i + \beta_i) < 1$ (Cryer and Chan, 2008). |
| Normality of innovations | – QQ plot; points should lie close to the line $y = x$,<br>– Jarque-Bera test; p-value less than the significance level suggests rejection of normality (Cryer and Chan, 2008). |
| Independence of standardized residuals | – Ljung-Box test; Q statistics; p-value exceeding the significant level indicates no serial correlation in residuals (Cryer and Chan, 2008). |

Table 3.5: Model diagnostics, GARCH($m, s$)

Figure 3.4a exhibits the ACF and PACF of log monthly returns of S&P 500 index. The ACF and PACF do not show significant correlations. Even at lags 4 and 12, the correlations are just mild. Therefore, the mean process (3.3a) does not include any AR or MA terms, and a white noise process seems plausible for $\delta_t$. Contrariwise, the ACF and PACF of the squared log returns, displayed in Figure 3.4b, admit significant autocorrelations and hence furnish the evidence for a higher-order serial dependence structure such as the GARCH model rather than a simple white noise process.

As mentioned earlier, to determine the GARCH orders $m$ and $s$, we need to identify the ARMA orders of the squared residuals $\varepsilon_t^2$. The extended autocorrelation function (EACF) proposed by Tsay and Tiao (1984) is utilized in order to effectively identify an ARMA model. In Table 3.6, the EACF suggests an ARMA(1,1) model for the squared residuals




(a) Log return

(b) Squared log return

Figure 3.4: Autocorrelation functions and partial autocorrelation functions, S&P 500

| AR\MA | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | × | × | × | × | ○ | ○ | ○ | ○ | × | ○ | ○ | ○ | ○ | ○ |
| 1 | × | ○ | ○ | ○ | ○ | ○ | ○ | ○ | × | ○ | ○ | ○ | ○ | ○ |
| 2 | × | × | ○ | ○ | ○ | ○ | ○ | ○ | × | ○ | ○ | ○ | ○ | ○ |
| 3 | × | × | × | ○ | ○ | ○ | ○ | ○ | × | ○ | ○ | ○ | ○ | ○ |
| 4 | × | × | × | × | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| 5 | × | × | × | × | × | ○ | ○ | ○ | × | ○ | ○ | ○ | ○ | ○ |
| 6 | × | × | ○ | × | × | × | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| 7 | ○ | × | ○ | ○ | × | × | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |

Table 3.6: Extended autocorrelation function, squared residuals

and accordingly a GARCH(1,1) model for $\sigma_t^2$.

The maximum likelihood estimation of aforementioned model, as returned by R{fGARCH}, is

$$\begin{aligned} \delta_t &= 6.509 \times 10^{-3} + \varepsilon_t, \\ \varepsilon_t &= \sigma_t \, z_t, \\ \sigma_t^2 &= 8.991 \times 10^{-5} + 0.1137\varepsilon_{t-1}^2 + 0.8408\sigma_{t-1}^2, \end{aligned} \tag{3.4}$$

where $z_t$ is Gaussian white noise with unit variance, and the conditional mean of $\delta_t$ is $6.509 \times 10^{-3}$.

Table 3.7 displays all related test outcomes. All parameter estimates are statistically significant. Parameters $\alpha_1$ and $\beta_1$ satisfy $\alpha_1 + \beta_1 < 1$, implying a stationary process which prevents volatilities from exploding. The Ljung-Box test accounts for the independence of the residuals. However, the normality of standardized residuals is strongly rejected, as suggested by the Jarque-Bera test result. Alternative evidence is provided by Figure 3.5. Historical data have a heavier tail than that implied by the model.

| **Test** | **Outputs** | |
|---|---|---|
| T-test on coefficients | $\mu$[5]: 1.07e-6 | $\alpha_0$: 0.0085 |
| (parameter: p-value) | $\alpha_1$: 6.77e-6 | $\beta_1$: <2e-16 |
| Jarque-Bera test (p-value) | 0 | |
| Ljung-Box test (p-value) | 0.4402[6] | |

Table 3.7: Test results, GARCH(1,1)

To improve the goodness of fit, one could assume that $z_t$ follows a heavy-tailed distribution such as a t-distribution. In this project, we use (3.4) as the model for the log monthly returns of the S&P 500 index. As in Section 3.1.1, we note the limitation of this model as a result of the residuals being imprecisely specified.

[5] $\mu$ represents the conditional mean of $\delta_t$

[6] The p-value corresponds to Q statistics at lag 15.

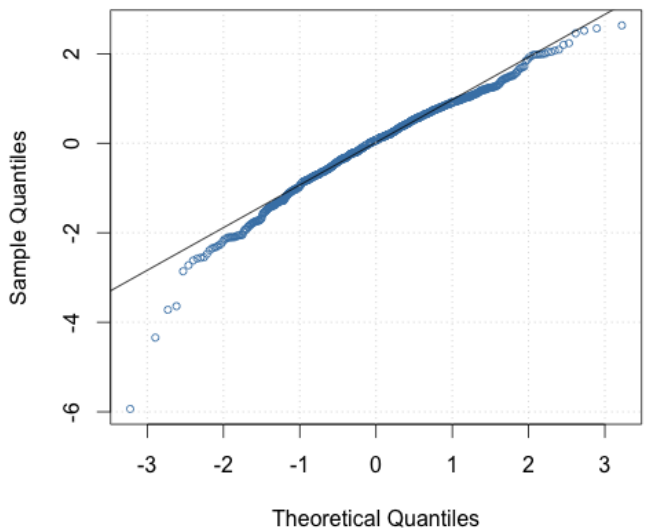



Figure 3.5: QQ plot of standardized residuals, S&P 500 vs GARCH(1,1)

## 3.2 Multivariate Model

Alternatively, we treat yields from the two assets systematically as a multivariate time series and utilize a vector autoregressive model (VAR) to capture their interaction. The VAR model is generalized from the univariate autoregressive model by letting all components be multidimensional. Suppose that a vector $\boldsymbol{y}_t$ at time $t$ comprises the interest rate of T-bills, $r_t$, and the effective rate of return of the S&P 500 index, denoted as $R_t$, both on a monthly basis. The returns of the S&P 500 index between Feb. 1950 and Dec. 1981 are ignored so that these two series are of the same length from Jan. 1982 to Sep. 2015. The two-dimensional VAR($p$) model is of the form

$$\boldsymbol{y}_t = \boldsymbol{\phi}_0 + \sum_{i=1}^{p} \boldsymbol{\phi}_i \boldsymbol{y}_{t-i} + \boldsymbol{\varepsilon}_t, \tag{3.5}$$

where $\boldsymbol{\phi}_0$ is a two-dimensional constant vector, the $\boldsymbol{\phi}_i$'s are $2 \times 2$ coefficient matrices, and $\boldsymbol{\varepsilon}_t$ is a two-dimensional white noise process with time-invariant positive definite covariance matrix $\boldsymbol{\Sigma}_\varepsilon \triangleq E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^T)$. An explicit presentation of the model is

$$\begin{pmatrix} R_t \\ r_t \end{pmatrix} = \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} + \sum_{i=1}^{p} \begin{pmatrix} \phi_{i,11} & \phi_{i,12} \\ \phi_{i,21} & \phi_{i,22} \end{pmatrix} \begin{pmatrix} R_{t-i} \\ r_{t-i} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}.$$

As in the univariate AR model, the lag length of the VAR model, $p$, can be selected based on information criteria. The general approach is to fit the VAR($p$) models with order $p = 0, 1, \ldots, p_{max}$ and choose the value of $p$ which minimizes the information criteria func-

tion. Three commonly used information criteria are Akaike (AIC), Schwarz-Bayesian (BIC) (Schwarz, 1980), and Hannan-Quinn (HQ) (Hannan and Quinn, 1979). By applying all three information criteria to a series of VAR models for T-bill rates and S&P 500 index returns, we obtain Figure 3.6 demonstrating how values of criteria functions vary as the order $p$ increases.

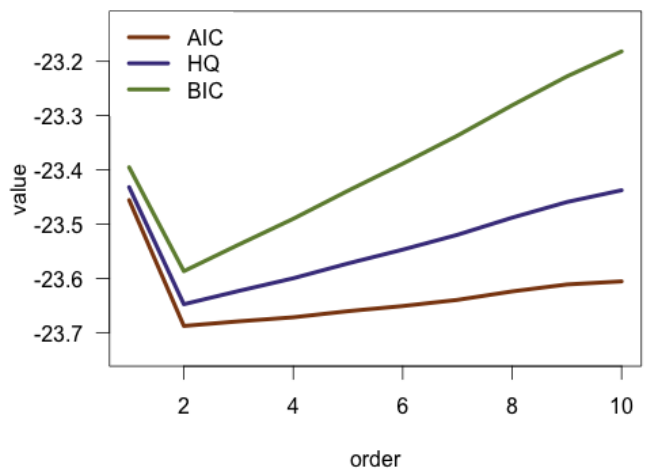



Figure 3.6: Information criteria, S&P 500 and T-bills

All of them drop to the minimum when $p = 2$. Therefore, we select the order $p = 2$. The least square estimates of the coefficients calculated by R{vars} are presented in Table 3.8.

| **Parameters** | **Estimates** | **P-values** |
|---|---|---|
| $\phi_{10}$ | 0.0047 | 0.206 |
| $\phi_{1,11}$ | 0.0389 | 0.437 |
| $\phi_{1,12}$ | $-12.2118$ | 0.183 |
| $\phi_{2,11}$ | $-0.0089$ | 0.860 |
| $\phi_{2,12}$ | 13.0142 | 0.152 |
| $\phi_{20}$ | $2.180e{-}5$ | 0.220 |
| $\phi_{1,21}$ | $6.578e{-}4$ | 0.007 |
| $\phi_{1,22}$ | 1.359 | <2e-16 |
| $\phi_{2,21}$ | $5.376e{-}4$ | 0.027 |
| $\phi_{2,22}$ | $-0.373$ | <2.76e-16 |

Table 3.8: Coefficient estimation, VAR(2)

To ensure that the fitted model is adequate and properly specified, model checking is con-

ducted to see if all fitted parameters are statistically significant, if there exist structural changes, if the process is stable, and if residuals violate the distributional assumptions, for example, independence and multivariate normality (Tsay, 2014). The significance of coefficients is examined per line by t-test as stated in Table 3.2 for the univariate ARMA models, and the resulting p-values are listed in Table 3.8 beside the parameter estimates. We see that none of coefficients determining the path dependence of $R_t$ are significantly non-zero. However, to preserve the relationship, we keep those parameters in the system. The VAR(2) model is estimated as

$$\begin{pmatrix} R_t \\ r_t \end{pmatrix} = \begin{pmatrix} 0.0047 \\ 2.180 \times 10^{-5} \end{pmatrix} + \begin{pmatrix} 0.0389 & -12.2118 \\ 6.578 \times 10^{-4} & 1.359 \end{pmatrix} \begin{pmatrix} R_{t-1} \\ r_{t-1} \end{pmatrix} + \begin{pmatrix} -0.0089 & 13.0142 \\ 5.376 \times 10^{-4} & -0.3731 \end{pmatrix} \begin{pmatrix} R_{t-2} \\ r_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \tag{3.6}$$

and $\boldsymbol{\Sigma}_{\varepsilon} = \begin{pmatrix} 1.883 \times 10^{-3} & 2.734 \times 10^{-7} \\ 2.734 \times 10^{-7} & 4.357 \times 10^{-8} \end{pmatrix}$.

With the back-shift operator $B$, (3.5) could be written as

$$\phi(B)\boldsymbol{y}_t = \boldsymbol{\phi}_0 + \boldsymbol{\varepsilon}_t,$$

where $\phi(B) = \boldsymbol{I}_2 - \sum_{i=1}^{p} \boldsymbol{\phi}_i B^i$ and $\boldsymbol{I}_2$ is an identity matrix of dimension two. Particularly for VAR(2), $\phi(B) = \boldsymbol{I}_2 - \boldsymbol{\phi}_1 B - \boldsymbol{\phi}_2 B^2$. The VAR($p$) (or VAR(2) specifically) is stationary when all solutions of the determinant equation $|\phi(B)| = 0$ exceed 1 in modulus (Tsay, 2014). The solutions of the equation system (3.6) are 0.9789, 0.2491, 0.2491, and 0.0605, suggesting stationarity of the estimated VAR(2) model given by (3.6).

The stability of the regression relationships in (3.6) can be assessed by the Cumulative Sum (CUSUM) test proposed by Barnard (1959). Results show that structural changes are barely feasible in a statistical sense (see Appendix A for details).

Recall that $\{\boldsymbol{\varepsilon}_t\}$ is a sequence of i.i.d. random vectors which follow a multivariate normal distribution with mean zero and positive-definite covariance matrix $\boldsymbol{\Sigma}_{\varepsilon}$. Table 3.9 summarizes the tests applied to the residuals to examine satisfaction of the requirements mentioned earlier and the corresponding results regarding (3.6).

Unfortunately, only the Portmanteau test gives a favorable outcome, confirming that residuals are independent over time. Evidence is against a constant variance-covariance matrix and normality of the residuals whose reasons are, to some extent, understood. In the uni-

| Properties | Tests | P-values |
|---|---|---|
| Heteroscedasticity | – ARCH test (Engle, 1982),<br>– p-value larger than the significance level indicates absence of heteroscedasticity. | <2.2e-16 |
| Serial and cross-sectional correlations | – multivariate Portmanteau test (Tsay, 2014),<br>– p-value exceeding the significance level indicates no correlations. | 0.5534 |
| Normality | – Jarque-Bera test (Jarque and Bera, 1980),<br>– p-value larger than the significance level indicates normality of residuals. | <2.2e-16 |

Table 3.9: Residual diagnostics, VAR(2)

variate model for the S&P 500 index (a constant conditional mean plus a GARCH(1,1) volatility process), the volatility process is statistically significant, implying the inherited variability of volatilities of S&P 500 returns. A simple model like VAR(2) which merely attributes changes in stock returns to the path dependence of the return itself would probably fail the test for constant variance. Moreover, the Jarque-Bera test of the univariate S&P 500 return model has pointed out that the normal distribution is not heavy-tailed enough to model the innovations hidden in the data. With the same data, the outcomes should be coherent, and thus normal innovation is still not an appropriate assumption even for VAR(2). However, for the sake of simplicity, we assume all basic assumptions of VAR(2) are satisfied and use (3.6) to emphasize the interdependence between the T-bill rates and the rates of return of the S&P 500 index.

# Chapter 4

# Numerical Analysis

In this chapter, we present the numerical results from the stochastic analysis. We first define sample contracts to be studied and discuss the values of the guaranteed maturity premium for such standard product designs. We then introduce the simulation procedure used in the numerical analysis. Results are shown afterwards with comments aiming to address questions such as how likely it is that the standard policies lapse, how much wealth the policyholder could anticipate from the policy, and how the corridor factors impact the death benefit.

## 4.1 Sample Contracts

For numerical illustrations, consider both Type A and Type B universal life insurance policies issued to a female nonsmoker aged 30 at time 0, with time measured in years. Both policies offer two investment account options: a savings account where the fund earns interest based on the yields of US 3-month T-bills and an equity account where returns track the performance of the S&P 500 index.

In terms of the basic savings account, the insurance company usually claims in the contract that higher interest rates may be credited and that the decision is made by the insurer. However, an attempt to model actual interest rate crediting behavior is challenging. The insurer's decision is subject to not only performance of the reference asset, but also a series of concerns such as competitiveness and stability of the returns and the financial position of the general asset portfolio. To simplify the problem, we proceed in the following way in this project. Firstly, no buffer is added to T-bill returns. Hence, credited rates of the savings account are strictly in line with the actual T-bill rates minus the management fee.

Secondly, nonnegative credit rates are guaranteed. This process is formalized as

$$i_{t-1}^S = \max(r_{t-1}^Y - f^S, 0),$$

where $i_{t-1}^S$ is the interest rate applied to the savings account in policy year $t$, $r_{t-1}^Y$ is the annually compounded T-bill rate in policy year $t$, and $f^S$ denotes the constant management fee for the savings account with $f^S = 0.1\%$ per annum.

The risky investment option (i.e., the equity account) is not eligible for any interest rate adjustment or guarantee as specified in the contract. The credited rate can be either negative or positive, merely depending on the annual rate of return of the S&P 500 index less the management fee; that is,

$$i_{t-1}^E = R_{t-1}^Y - f^E,$$

where $i_{t-1}^E$ is the interest rate credited to the equity account in policy year $t$, $R_{t-1}^Y$ is the annual rate of return of the S&P 500 index in policy year $t$, and $f^E$ represents the management fee for the equity account with $f^E = 1.67\%$ per annum[1].

Suppose that the insured is rated as a standard life, i.e., $\kappa = 1$, and her mortality rate $q_x$ is estimated by the 2001 CSO Ultimate Table[2]. When the fund is allocated to both accounts, the COI charge is first subtracted from the savings account until no balance remains. The amount due (if any) then reduces the value of the equity account until no balance is left. The policy lapses when both accounts are exhausted. The policies, for both Type A and Type B, are assumed to be paid up at age 65. The face amount, $FA$, is \$100000, and the expense charge rate $e$ is assumed 5% of the premium deposit. The death benefit discount factor $v_{t-1}$ for policy year $t$ is equivalent to the applicable credited rate, i.e.,

$$v_{t-1} = (1 + i_{t-1})^{-1}.$$

The corridor factor $c_x$ is from Lombardi (2006) (see Appendix B for the list of values).

## 4.2 Guaranteed Maturity Premium

The guaranteed maturity premium (GMP), $P^G$, is calculated under a deterministic framework. We presume the most conservative circumstances where all premiums are placed in the savings account. Table 4.1 shows GMPs of Type A and Type B policies at various levels of the guaranteed credited rate $i^G$.

[1]The values of $f^S$ and $f^E$ are chosen based on Manulife's sample contract with adjustment. http://www.manulife.ca

[2]The table is from 2001 Commissioners Standard Ordinary (CSO) Task Force Report.

| GMP $i^G$ / Type | 0% | 1% | 1.5% | 2% | 2.5% | 8% |
|---|---|---|---|---|---|---|
| Type A | 3342.76 | 2148.02 | 1821.53 | 1549.82 | 1323.42 | 320.26 |
| Type B | 47241.56 | 25806.02 | 19125.31 | 14207.44 | 10581.90 | 678.48 |

Table 4.1: Guaranteed maturity premium

The GMPs of both types decrease as the guaranteed credited rate increases, but the change is realized through mechanisms differing by product types. On the guaranteed basis, the net premium (premiums after expense deductions) and accrued interest exactly cover all insurance charges as discussed in Section 2.3. For a Type B policy, the NAAR remains constant over time. Then according to (2.2), the COI charges vary in a pattern exclusively determined by mortality rates. In other words, the value of the guaranteed credited rate does not impact COI charges. However, a large $i^G$ speeds up account value accumulation, which enables the policyholder to pay off the insurance charges with less premiums. In contrast, the NAAR of a Type A policy decreases as the account value is being built up since the sum, or the death benefit, is fixed. This results in a downward trend of COI charges even without a high guaranteed credited rate. With high guaranteed credited rates, the shrinking NAAR and the accelerating interest accrual cooperate in dragging the GMP down.

Note that the guaranteed maturity premium is the maximum that the policyholder needs to pay in order to prevent policy lapse. Theoretically it only applies to the savings account because the equity account is not designed with an interest rate guarantee. For the equity account, the GMP is more like an approximate premium at a certain interest rate level.

## 4.3 Simulation Framework

It would have been the best outcomes if we could establish the probability distribution of the account value $AV_t$ from (2.8). However, recursive substitution of the $AV$'s gets complicated especially for Type A products whose $NAAR$ varies with the $AV$ as well, and for both types the $AV$ is obviously highly dependent on the path of $\{i_t\}$. The path dependence of the policyholder's account value and the complex relationship between the net amount at risk and the account value makes it difficult to find analytical expressions for the $AV$. Therefore, simulation analysis is conducted to throw light on potential future cash flow outcomes.

The steps involved in a single scenario are summarized as follows.

- Simulate ($\omega-$issue age)$\times$12 log effective T-bill rates and forces of interest on the S&P 500 index at monthly frequency from the univariate model in Section 3.1 and convert to effective rates. Generate ($\omega-$issue age)$\times$12 monthly effective rates for both assets from the multivariate model in Section 3.2. Results include two sequences for the T-bills (one from the univariate model and the other from the multivariate model) and two sequences for the S&P 500 index.

- Compound to annual interest rates. Each sequence is accordingly compressed to length of ($\omega-$issue age).

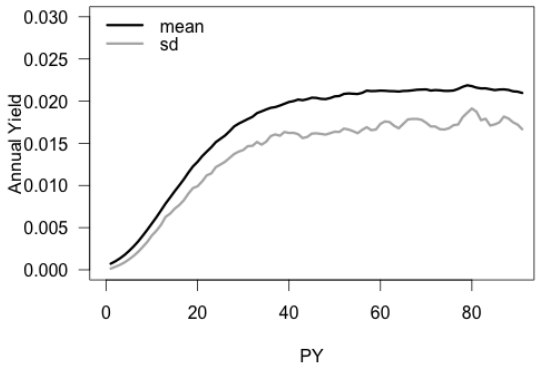
- Given product type, premium payment plan, and allocation strategy, let the fund grow along the simulated paths of the univariate and multivariate models according to the mechanism discussed in Chapter 2. Two sets of future cash flows are obtained, including expense charge, cost of insurance, credited interest, and account value in any policy year between 0 and ($\omega-$issue age). Each set contributes to one potential outcome under the univariate and multivariate models, respectively.

In total 5000 scenarios are generated. Hence the empirical distribution of every element of cash flows is of size $n = 5000$.

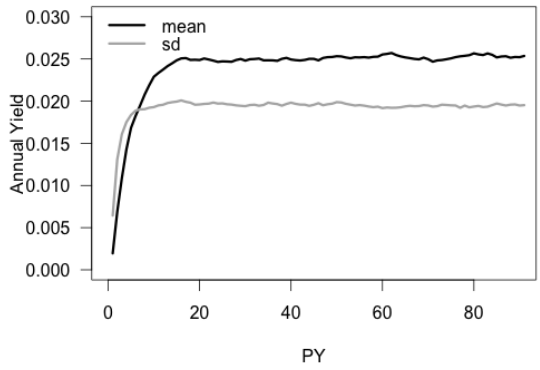
## 4.4 Cash Flow Projections

### 4.4.1 Simulated rates of return

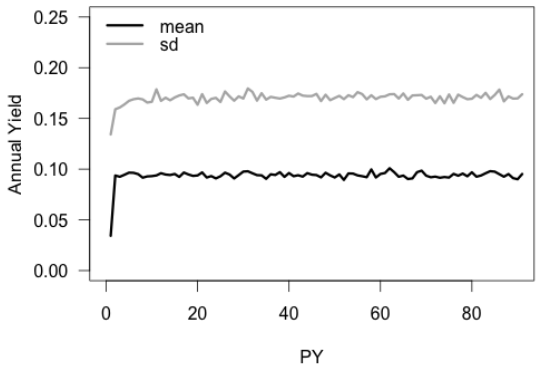
Figure 4.1 illustrates where the average annual interest rates lands and how much they variate. By comparing Figures 4.1c and 4.1d, we see that the mean of the S&P 500 returns simulated from the univariate model are a bit higher than those from the multivariate model. Meanwhile, the GARCH process expands variability of the annual rates through fluctuating volatilities. Overall, the long-term average for both models are close to the mean of the historical annual rates of return. Opposite conclusions are reached from the comparison between Figures 4.1a and 4.1b. Despite the different speeds of convergence, the long-term levels predicted by the two models are consistent. In the long run, the mean of the T-bill returns based on the univariate model are slightly lower than those from the multivariate model. But neither approaches the level suggested by the historical mean. Recall that both models violate the assumption regarding the normality of residuals and that historical data suggest residuals be modelled by a heavy-tailed distribution. Therefore, the standard deviations of the annual rates are understated by the current two asset models.



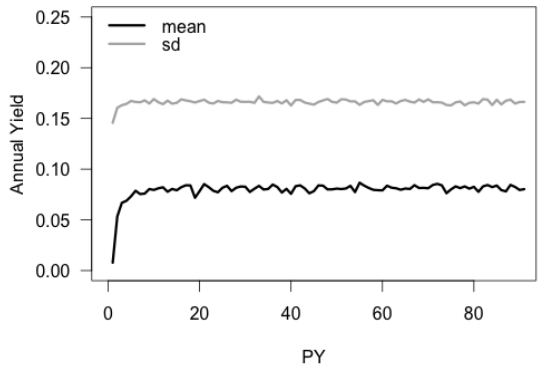

(a) T-bill, univariate model




(b) T-bill, multivariate model




(c) S&P 500, univariate model




(d) S&P 500, multivariate model

Figure 4.1: Mean and standard deviation of annually compounded simulated rates of return

### 4.4.2 Premium payment and allocation strategies

Another essential step is to forecast the amount of future premiums which the policyholder usually decides in his or her best interest. Above all, the accumulated value should be no less than the sum of all costs. Apparently, the S&P 500 index generates higher returns than T-bills, so less deposits into the equity account would meet the requirement. For instance, if the credited rates applied to the savings account and the equity account are constants estimated by the process mean of (3.2) (equivalent to an annual effective rate of 1.62%) and the conditional mean of (3.4) (corresponding to an annual effective rate of 8.12%) , respectively, the GMP of a Type A contract, judging from Table 4.1, is roughly between \$1550 and \$1821 when totally invested in the savings account, and the GMP drops to about \$320 when the fund is completely invested in the S&P 500 index. For comparison purposes, we suppose that the policyholder pays \$1700 each year (correspond to a 1.71% interest rate) during the payment period for a Type A contract regardless of the policyholder's investment strategies and fund performance. The annual premium for a Type B contract is assumed to be \$15000 (approximately correspond to 1.91%).

To reflect different investment needs of policyholders, we consider four premium allocation strategies: $w = 0\%$, 40%, 60%, and 100% with $w$ denoting the proportion of premium placed in the savings account.

### 4.4.3 Policy lapse

#### Type A policy

Table 4.2 lists the numbers of scenarios where the Type A policies are in force at various time points. Recall that the premium payment ends in the 35th policy year. After that, the policyholder counts on the wealth in the account rather than making additional deposits to withstand bad investment outcomes, increasing insurance charges, or both. This action puts the insured in danger of losing the coverage if deposits made during the payment period are inadequate, and fluctuations in stochastic asset returns may aggravate or improve the situation depending on their direction. From Table 4.2, we see that lapses occur under all premium allocation strategies regardless of the asset model used and that they all happen after the 40th policy year. In contrast, Figure 4.3, which illustrates the development of account values under a deterministic framework (see Section 4.4.4 for explanations), indicates that the policy will not lapse for sure unless $w = 1$. It is because the premium we suppose needs an interest rate of 1.71% to prevent the policy from lapse while in the deterministic analysis the fund is growing by 1.62%. Table 4.2 and Figure 4.3 are consistent in a sense that the most lapses appear when $w = 1$. On the other hand, there are "survivors" even

after the 70th policy year when $w = 1$, suggesting the notable impact of "good years".

| Policy year | Attained Age | Univariate model | | | | Multivariate model | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ |
| 20 | 50 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 |
| 40 | 70 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 |
| 50 | 80 | 4999 | 5000 | 5000 | 5000 | 4972 | 4998 | 5000 | 5000 |
| 70 | 100 | 4911 | 4931 | 4923 | 1804 | 4573 | 4683 | 4747 | 4410 |
| 91 | 121 | 4897 | 4905 | 4897 | 1610 | 4484 | 4594 | 4674 | 4261 |

Table 4.2: Numbers of in-force policies, Type A

Secondly, opposite signals are received from the comparison between two asset models: under the univariate model, the policyholder is safer when the fund is, partially or totally, invested in the equity account and riskier when the fund is dumped in the savings account. This is probably because on average the univariate model gives higher S&P 500 returns but lower T-bill returns. The slow convergence of T-bill rates under the univariate model, as demonstrated in Figure 4.1a, contributes to the fund missing the chance of early accumulation, which results in increased lapse probabilities in later years.

Notice that regardless of the asset model chosen, a conservative investment strategy where all premiums are deposited into the savings account (i.e., $w = 1$) is least effective in keeping the policy in force, due to the absolute low returns. The other extreme ($w = 0$) is better because high returns offset the potential loss incurred by frequent fluctuations. The best choice is a "mixed" or "balanced" strategy allocating premiums to both accounts. Though the optimal $w$ is not discussed, balanced premium allocation strategies overall do secure the policy against lapse to the largest extent.

**Type B policy**

Analysis for a Type B product is conducted in the same vein, and results are shown in Table 4.3. Some similar conclusions are drawn. Here we only bring up phenomena or features specific to Type B contracts.

By comparing Table 4.3 and Figure 4.5 where the development of account values under a deterministic framework is illustrated (see Section 4.4.4 for details), we see that the stochastic analysis detects possible policy lapses when $w = 0.4$ or 0.6 at the timing which is roughly consistent with the deterministic projection. Moreover, when $w = 1$, the policy can not even stay in force after the 60th policy year under the deterministic analysis because the constant credited rate assumed fails to support the premium amount we suppose. But the varying T-bill rates generated from the stochastic models extend the life of the contract

at least to 70 years with the most probability.

| Policy year | Attained Age | Univariate model | | | | Multivariate model | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ |
| 20 | 50 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 |
| 40 | 70 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 |
| 50 | 80 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 | 5000 |
| 70 | 100 | 5000 | 5000 | 5000 | 5000 | 4983 | 5000 | 5000 | 5000 |
| 91 | 121 | 4960 | 4959 | 4954 | 1651 | 4720 | 4757 | 4782 | 4166 |

Table 4.3: Numbers of in-force policies, Type B

### 4.4.4 Projected account values

#### Type A policy

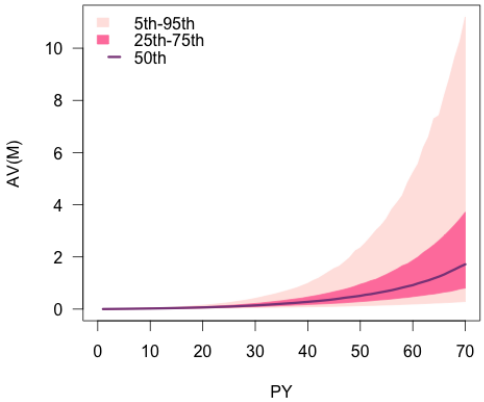
Figure 4.2 presents the account value projections with stochastic asset returns. Each subfigure demonstrates a projection based on one combination of premium allocation strategies ($w$ varying among 0, 0.4, 0.6, and 1) and asset models (univariate or multivariate). For each projection, the development of various percentiles of the account value is depicted on the left, and values of key statistics are listed on the right. Only scenarios that keep the policy in force till the limiting age are considered[3]. Account values are forecast under a deterministic framework where the annual T-bill rate of return is 1.62% (equivalent to the long-term level estimated by (3.2)[4] ) and the annual S&P 500 rate of return is 8.12% (equivalent to the conditional mean in monthly scale from (3.4)[5]); Figure 4.3 displays the growth of the account value on the left and key values on the right.

Except for Figures 4.2d and 4.2h, all plots are in a similar shape. The percentiles move upwards as the fund is being built up. The distribution of the account values at each time point is skewed to the right. The right tail gets fatter as years pass, and tremendous upside potential is created in later years. Figures 4.2h and 4.2d are alike: neither the skewness of the distribution nor growth in the account value is as striking as in the others.

[3]In order to compare account values over time, scenarios that lead to policy lapses at any time are excluded. However, from the insurer's standpoint, this measure overstates potential account values.

[4]The annual rate is calculated by $(e^{-6.6157} + 1)^{12} - 1$.

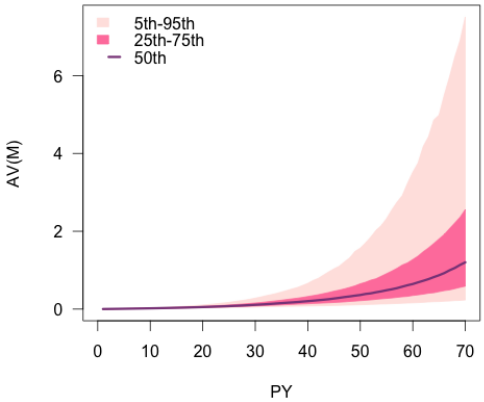
[5]The annual rate is calculated by $e^{12*6.509\times10^{-3}} - 1$




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0311 | 0.0901 | 0.1320 | 0.2898 |
| 25th percentile | 0.0478 | 0.1758 | 0.2870 | 0.8061 |
| 50th percentile | 0.0640 | 0.2780 | 0.5089 | 1.7162 |
| 75th percentile | 0.0890 | 0.4620 | 0.9510 | 3.7144 |
| 95th percentile | 0.1481 | 0.9827 | 2.3460 | 11.2003 |
| Mean | 0.0737 | 0.3810 | 0.7894 | 3.3292 |
| SD | 0.0399 | 0.3698 | 0.9109 | 5.4899 |

* out of 4897 scenarios

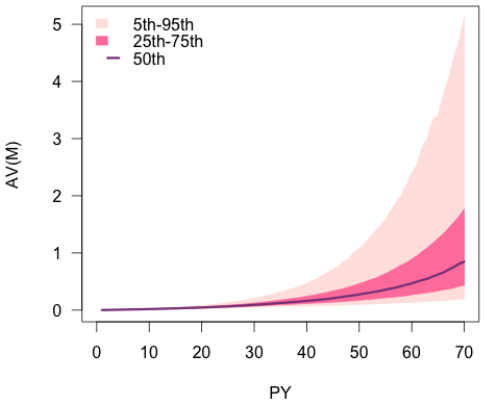
(a) $w = 0$, univariate model




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0315 | 0.0810 | 0.1126 | 0.2348 |
| 25th percentile | 0.0423 | 0.1384 | 0.2164 | 0.5888 |
| 50th percentile | 0.0524 | 0.2046 | 0.3622 | 1.2015 |
| 75th percentile | 0.0683 | 0.3229 | 0.6502 | 2.5562 |
| 95th percentile | 0.1053 | 0.6595 | 1.5664 | 7.5088 |
| Mean | 0.0585 | 0.2709 | 0.5457 | 2.2726 |
| SD | 0.0251 | 0.2396 | 0.5979 | 3.6457 |

* out of 4905 scenarios

(b) $w = 0.4$, univariate model

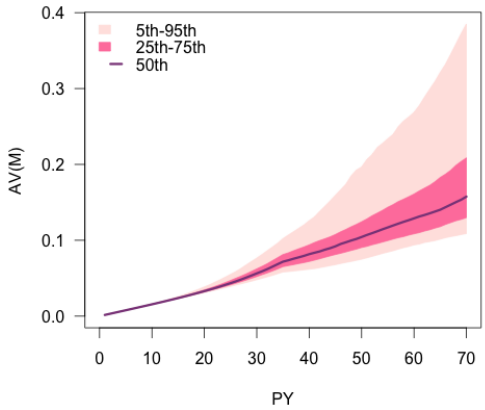



| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0319 | 0.0764 | 0.1016 | 0.1970 |
| 25th percentile | 0.0390 | 0.1169 | 0.1737 | 0.4359 |
| 50th percentile | 0.0459 | 0.1615 | 0.2735 | 0.8476 |
| 75th percentile | 0.0565 | 0.2410 | 0.4627 | 1.7668 |
| 95th percentile | 0.0814 | 0.4662 | 1.0746 | 5.1112 |
| Mean | 0.0500 | 0.2053 | 0.3942 | 1.5779 |
| SD | 0.0169 | 0.1602 | 0.4000 | 2.4717 |

* out of 4897 scenarios

(c) $w = 0.6$, univariate model

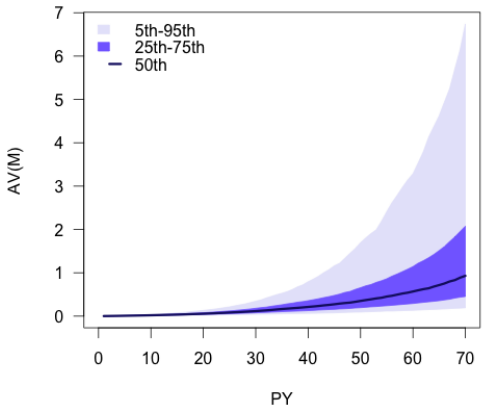
Figure 4.2: Percentiles, mean, and standard deviation of account value, Type A, in million $




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0310 | 0.0619 | 0.0751 | 0.1089 |
| 25th percentile | 0.0320 | 0.0721 | 0.0905 | 0.1298 |
| 50th percentile | 0.0329 | 0.0817 | 0.1043 | 0.1576 |
| 75th percentile | 0.0345 | 0.0939 | 0.1244 | 0.2085 |
| 95th percentile | 0.0385 | 0.1258 | 0.1966 | 0.3852 |
| Mean | 0.0337 | 0.0867 | 0.1154 | 0.1914 |
| SD | 0.0027 | 0.0260 | 0.0481 | 0.1262 |

* out of 1610 scenarios

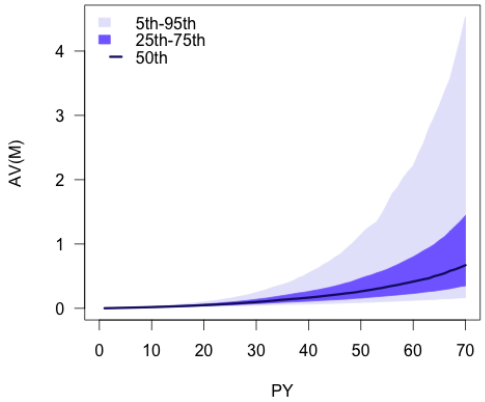
(d) $w = 1$, univariate model




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0258 | 0.0648 | 0.0943 | 0.1928 |
| 25th percentile | 0.0403 | 0.1277 | 0.1954 | 0.4529 |
| 50th percentile | 0.0557 | 0.2124 | 0.3510 | 0.9308 |
| 75th percentile | 0.0779 | 0.3557 | 0.6553 | 2.0733 |
| 95th percentile | 0.1319 | 0.7926 | 1.6924 | 6.7424 |
| Mean | 0.0640 | 0.2933 | 0.5650 | 2.0788 |
| SD | 0.0358 | 0.2831 | 0.8326 | 10.4227 |

* out of 4484 scenarios

(e) $w = 0$, multivariate model

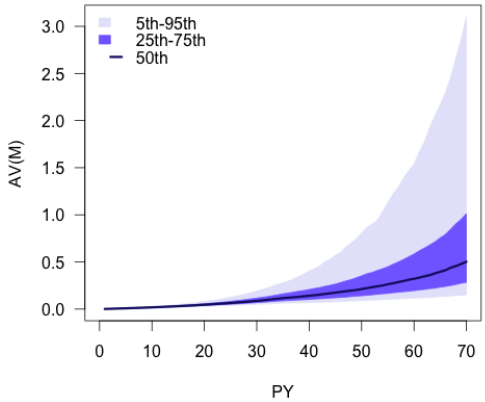



| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0291 | 0.0671 | 0.0920 | 0.1675 |
| 25th percentile | 0.0391 | 0.1114 | 0.1595 | 0.3492 |
| 50th percentile | 0.0494 | 0.1660 | 0.2639 | 0.6708 |
| 75th percentile | 0.0636 | 0.2615 | 0.4681 | 1.4423 |
| 95th percentile | 0.0986 | 0.5490 | 1.1466 | 4.5323 |
| Mean | 0.0546 | 0.2196 | 0.4042 | 1.4371 |
| SD | 0.0232 | 0.1852 | 0.5469 | 6.8236 |

* out of 4594 scenarios

(f) $w = 0.4$, multivariate model

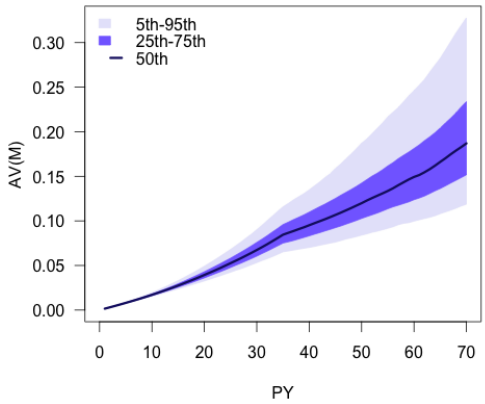
Figure 4.2: Percentiles, mean, and standard deviation of account value, Type A, in million $ (con't)




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0308 | 0.0699 | 0.0904 | 0.1517 |
| 25th percentile | 0.0384 | 0.1030 | 0.1412 | 0.2842 |
| 50th percentile | 0.0460 | 0.1414 | 0.2127 | 0.5023 |
| 75th percentile | 0.0562 | 0.2079 | 0.3526 | 1.0139 |
| 95th percentile | 0.0799 | 0.4030 | 0.8049 | 3.1002 |
| Mean | 0.0495 | 0.1770 | 0.3070 | 1.0134 |
| SD | 0.0164 | 0.1262 | 0.3675 | 4.5629 |

* out of 4674 scenarios

(g) $w = 0.6$, multivariate model

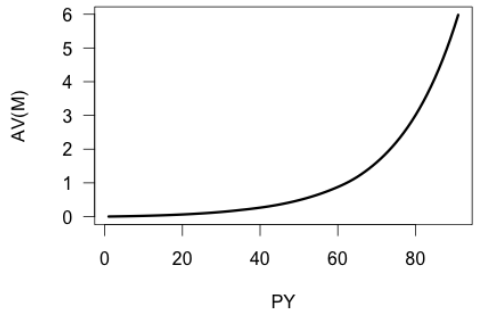



| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.0327 | 0.0703 | 0.0840 | 0.1188 |
| 25th percentile | 0.0361 | 0.0832 | 0.1028 | 0.1517 |
| 50th percentile | 0.0394 | 0.0951 | 0.1199 | 0.1870 |
| 75th percentile | 0.0429 | 0.1097 | 0.1423 | 0.2337 |
| 95th percentile | 0.0491 | 0.1347 | 0.1870 | 0.3278 |
| Mean | 0.0399 | 0.0980 | 0.1254 | 0.2008 |
| SD | 0.0051 | 0.0203 | 0.0319 | 0.0684 |

* out of 4261 scenarios

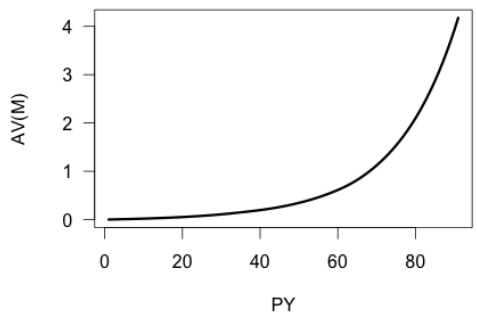
(h) $w = 1$, multivariate model

Figure 4.2: Percentiles, mean, and standard deviation of account value, Type A, in million $ (con't)



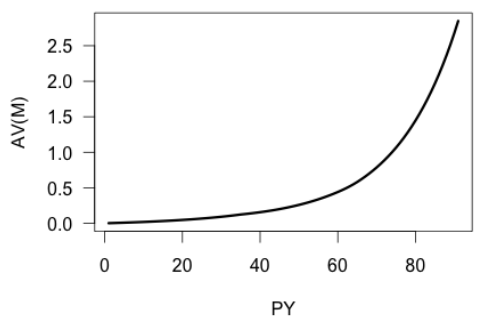

| Policy year | Account value |
|---|---|
| 20 | 0.0632 |
| 40 | 0.2654 |
| 50 | 0.4867 |
| 70 | 1.6082 |

(a) $w = 0$



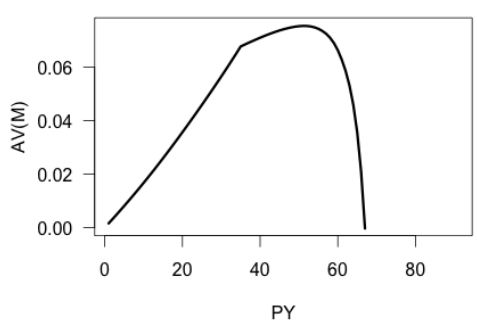

| Policy year | Account value |
|---|---|
| 20 | 0.0530 |
| 40 | 0.1969 |
| 50 | 0.3461 |
| 70 | 1.1225 |

(b) $w = 0.4$




| Policy year | Account value |
|---|---|
| 20 | 0.0472 |
| 40 | 0.1561 |
| 50 | 0.2591 |
| 70 | 0.7862 |

(c) $w = 0.6$




| Policy year | Account value |
|---|---|
| 20 | 0.0357 |
| 40 | 0.0711 |
| 50 | 0.0754 |
| 70 | lapse |

(d) $w = 1$

Figure 4.3: Development of account value with deterministic returns, Type A, in million $

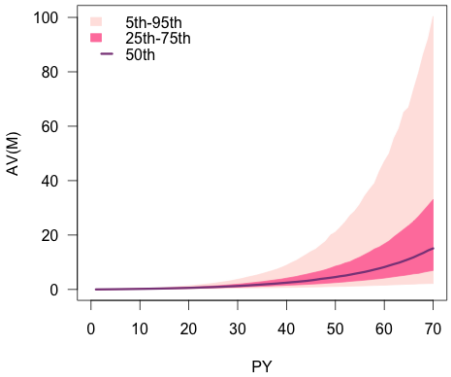
An immediate conclusion supported by Figure 4.2 is that an aggressive investment strategy normally creates more wealth at a price. The means and the standard deviations of the account value increase as more fund is invested in the equity account. As a measure of costs or risks, the standard deviation quantifies the extent of variability while the 5th percentile quantifies the worst scenario with certain confidence. Larger standard deviations indicate more extreme investment outcomes on both ends. The increase in upside is not our concern but the decrease in downside is. However, a risky portfolio does not necessarily lead to deterioration of downside potential. We see that except in the 20th policy year when $w$ is 0 or 0.4 under the univariate model and the 20th and 40th policy years when $w$ is 0, 0.4, or 0.6 under the multivariate model, the 5th percentiles increase as $w$ decreases. Namely, in the majority of time, the worst outcome is improved by risky premium allocation strategies. In this regard, despite the security provided by the guaranteed credited rate, the advantage of the savings account is limited from the policyholder's perspective. The policyholder could bet on a risky investment strategy generating more wealth without sacrificing account values. Combining the results from Table 4.2, a wise choice would be a moderate premium allocation strategy which maintains the coverage effectively and accommodate to one's investment needs.

As to the difference across asset models, the interdependence between the returns of two investment options makes a difference in account value projections. We see that the means of the account value under the univariate model are higher than their counterparts under the multivariate model as long as the fund is (partially or fully) placed in the equity account. But the relation reverses when the savings account is the only chosen investment option.

The contrast between Figures 4.2 and 4.3 shows that the mean account values projected under the stochastic framework considerably surpass the corresponding deterministic projections. But in most instances, the medians (i.e., the 50th percentiles) of the account value projected stochastically are closer to their counterparts under the deterministic analysis. We could expect closer gaps if studying the means and medians without excluding the scenarios that render the policy lapse during the coverage period. We can not assert which projection is superior but admit that the stochastic analysis gives more information on future account values.

**Type B policy**

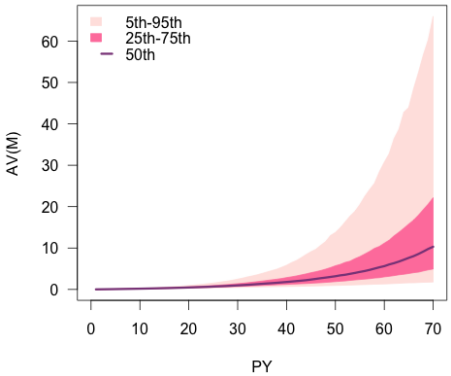
The stochastic and deterministic projection results are displayed in Figures 4.4 and 4.5 in the same format as for Type A policies.




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2833 | 0.7865 | 1.1126 | 2.2250 |
| 25th percentile | 0.4329 | 1.5695 | 2.5380 | 7.0029 |
| 50th percentile | 0.5786 | 2.4864 | 4.5572 | 15.0955 |
| 75th percentile | 0.8024 | 4.1451 | 8.5400 | 33.0307 |
| 95th percentile | 1.3350 | 8.8646 | 21.0449 | 100.3841 |
| Mean | 0.6654 | 3.4131 | 7.0634 | 29.6264 |
| SD | 0.3591 | 3.3301 | 8.2011 | 49.3678 |

* out of 4960 scenarios

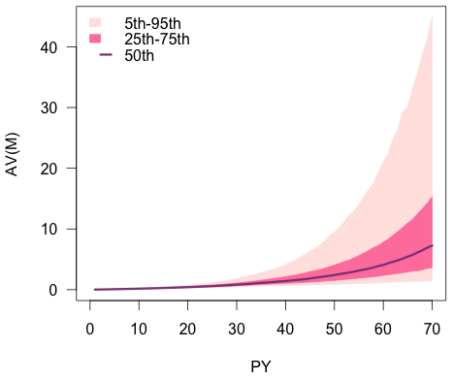
(a) $w = 0$, univariate model




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2898 | 0.7258 | 0.9803 | 1.8006 |
| 25th percentile | 0.3818 | 1.2228 | 1.8931 | 4.9359 |
| 50th percentile | 0.4699 | 1.8029 | 3.1852 | 10.3133 |
| 75th percentile | 0.6085 | 2.8500 | 5.7062 | 22.1461 |
| 95th percentile | 0.9328 | 5.8185 | 13.7564 | 65.9746 |
| Mean | 0.5237 | 2.3881 | 4.7877 | 19.7339 |
| SD | 0.2197 | 2.1063 | 5.2602 | 32.0623 |

* out of 4959 scenarios

(b) $w = 0.4$, univariate model

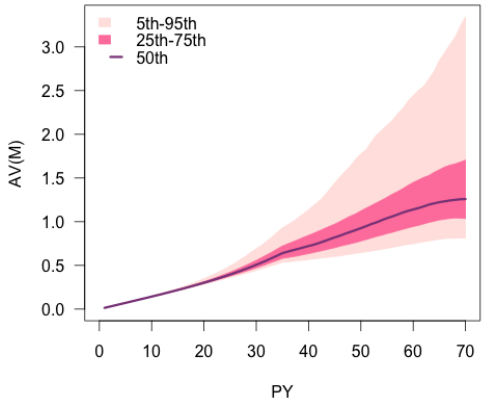



| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2934 | 0.6928 | 0.8930 | 1.4943 |
| 25th percentile | 0.3548 | 1.0384 | 1.5326 | 3.6336 |
| 50th percentile | 0.4139 | 1.4287 | 2.4039 | 7.2655 |
| 75th percentile | 0.5065 | 2.1259 | 4.0620 | 15.2974 |
| 95th percentile | 0.7236 | 4.0901 | 9.4376 | 44.8726 |
| Mean | 0.4499 | 1.8172 | 3.4661 | 13.6577 |
| SD | 0.1469 | 1.4070 | 3.5171 | 21.7258 |

* out of 4954 scenarios

(c) $w = 0.6$, univariate model

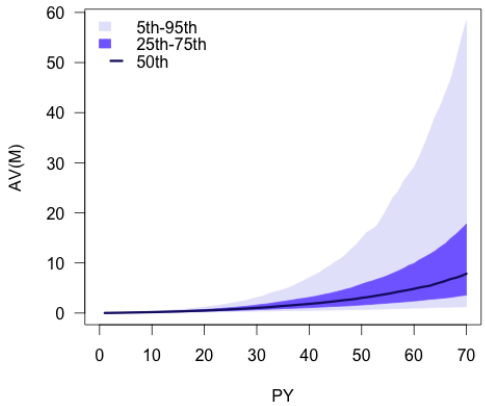
Figure 4.4: Percentiles, mean, and standard deviation of account value, Type B, in million $




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2855 | 0.5672 | 0.6475 | 0.8149 |
| 25th percentile | 0.2931 | 0.6356 | 0.7792 | 1.0371 |
| 50th percentile | 0.3012 | 0.7216 | 0.9263 | 1.2591 |
| 75th percentile | 0.3138 | 0.8437 | 1.1232 | 1.7038 |
| 95th percentile | 0.3476 | 1.1340 | 1.7752 | 3.3434 |
| Mean | 0.3072 | 0.7735 | 1.0221 | 1.5698 |
| SD | 0.0227 | 0.2334 | 0.4371 | 1.1337 |

* out of 1651 scenarios

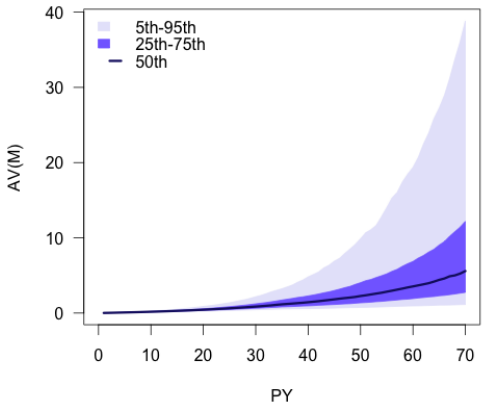
(d) $w = 1$, univariate model




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2304 | 0.5393 | 0.7272 | 1.2745 |
| 25th percentile | 0.3610 | 1.0814 | 1.6213 | 3.5971 |
| 50th percentile | 0.4978 | 1.8230 | 3.0319 | 7.8515 |
| 75th percentile | 0.6933 | 3.1146 | 5.7465 | 17.7445 |
| 95th percentile | 1.1788 | 7.0382 | 14.8656 | 58.3253 |
| Mean | 0.5723 | 2.5595 | 4.9055 | 17.8621 |
| SD | 0.3203 | 2.5274 | 7.3851 | 91.7227 |

* out of 4720 scenarios

(e) $w = 0$, multivariate model

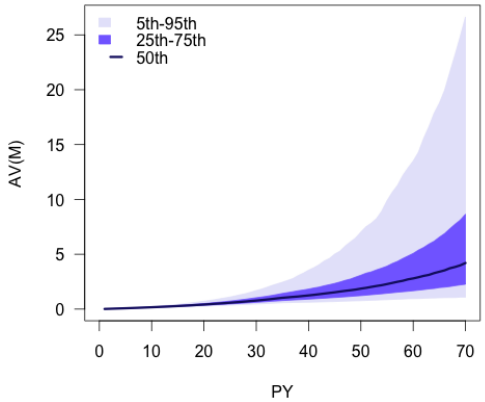



| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2663 | 0.5956 | 0.7459 | 1.1399 |
| 25th percentile | 0.3526 | 0.9573 | 1.3617 | 2.7475 |
| 50th percentile | 0.4421 | 1.4396 | 2.2670 | 5.5853 |
| 75th percentile | 0.5666 | 2.2763 | 4.0382 | 12.1875 |
| 95th percentile | 0.8724 | 4.8085 | 9.8816 | 38.8453 |
| Mean | 0.4880 | 1.9097 | 3.4819 | 12.1404 |
| SD | 0.2031 | 1.6187 | 4.7608 | 59.0562 |

* out of 4757 scenarios

(f) $w = 0.4$, multivariate model

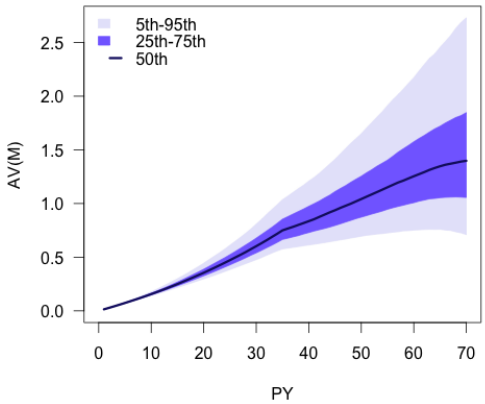
Figure 4.4: Percentiles, mean, and standard deviation of account value, Type B, in million $ (con't)




| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2836 | 0.6272 | 0.7691 | 1.0895 |
| 25th percentile | 0.3492 | 0.9079 | 1.2263 | 2.2632 |
| 50th percentile | 0.4153 | 1.2467 | 1.8663 | 4.2039 |
| 75th percentile | 0.5033 | 1.8231 | 3.0873 | 8.6493 |
| 95th percentile | 0.7113 | 3.5461 | 7.0228 | 26.6404 |
| Mean | 0.4453 | 1.5593 | 2.6797 | 8.6230 |
| SD | 0.1432 | 1.1071 | 3.2135 | 39.7097 |

* out of 4782 scenarios

(g) $w = 0.6$, multivariate model

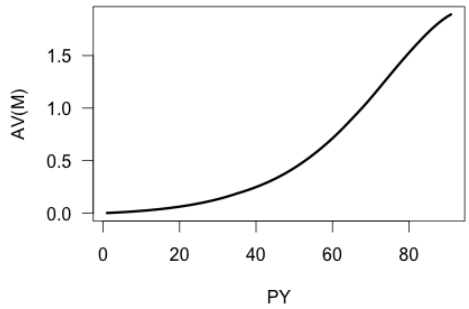



| Policy year | 20 | 40 | 50 | 70 |
|---|---|---|---|---|
| 5th percentile | 0.2987 | 0.6139 | 0.6958 | 0.7113 |
| 25th percentile | 0.3271 | 0.7295 | 0.8733 | 1.0563 |
| 50th percentile | 0.3548 | 0.8355 | 1.0416 | 1.3979 |
| 75th percentile | 0.3875 | 0.9735 | 1.2537 | 1.8462 |
| 95th percentile | 0.4429 | 1.2075 | 1.6481 | 2.7287 |
| Mean | 0.3643 | 0.8918 | 1.1381 | 1.6422 |
| SD | 0.0453 | 0.1832 | 0.2884 | 0.6208 |

* out of 4166 scenarios

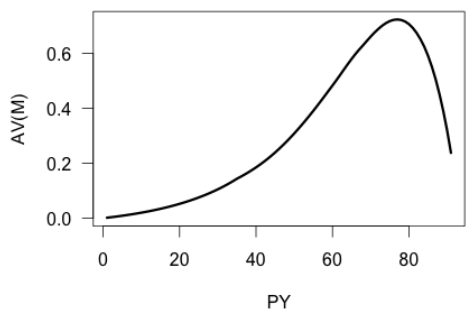
(h) $w = 1$, multivariate model

Figure 4.4: Percentiles, mean and standard deviation of account value, Type B, in million $ (con't)



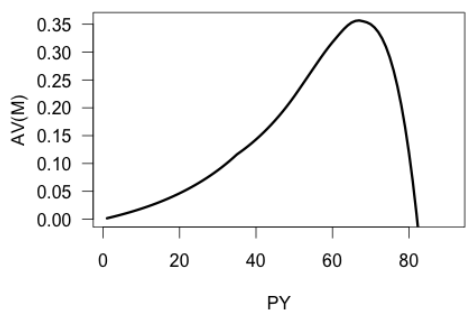

| Policy year | Account value |
|---|---|
| 20 | 0.0621 |
| 40 | 0.2483 |
| 50 | 0.4302 |
| 70 | 1.0943 |

(a) $w = 0$



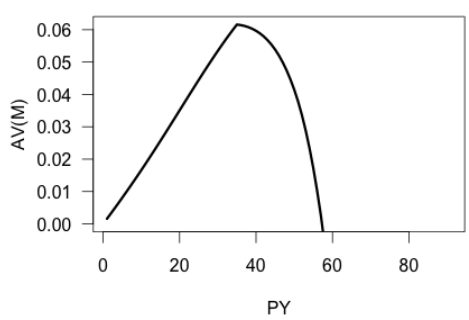

| Policy year | Account value |
|---|---|
| 20 | 0.0522 |
| 40 | 0.1849 |
| 50 | 0.3085 |
| 70 | 0.6612 |

(b) $w = 0.4$




| Policy year | Account value |
|---|---|
| 20 | 0.0465 |
| 40 | 0.1432 |
| 50 | 0.2206 |
| 70 | 0.3500 |

(c) $w = 0.6$




| Policy year | Account value |
|---|---|
| 20 | 0.0351 |
| 40 | 0.0597 |
| 50 | 0.0414 |
| 70 | lapse |

(d) $w = 1$

Figure 4.5: Development of account value with deterministic returns, Type B, in million $

In Figure 4.4, the percentiles in most plots increase following a familiar trend. But a unique growth pattern shows up in Figures 4.4d and 4.4h: the percentiles reach a peak and then drop. It is especially obvious for the 5th and 25th percentiles. The mechanism of Type B policies accounts for this special feature. At the early stages, net deposits (excluding the expense charge) and interest are more than enough to pay the COI charges, and the excess contributes to fund accumulation. For a Type B contract, the COI charge increases as the insured ages without the account value offsetting the NAAR. In later years the COI could become so large that inflows do not cover it. The deficit needs to be made up by withdrawals from one or both accounts. Thus the account value starts to decline. Conversely, the 95th percentile does not have a reversal point. Hence, if the returns are favorable enough, it is still possible that interest could cover the increasing COIs. Then it is not necessary to lose account values.

### 4.4.5 Impact from corridor factors

#### Type A policy

Table 4.4 gives the numbers of scenarios where the death benefit is the amount required by corridor factors. First, all scenarios that keep Type A policies in force throughout the insurance period also raise the amount of death benefit. This is because the interest earned in those scenarios is substantial so that the policy could "survive" all future deductions even without deposits after the 35th policy year. Then the proportion of the savings component exceeds the threshold set up by the corridor factors. The death benefit is augmented accordingly to ensure the insurance function of the policy.

| Policy year | Attained Age | Univariate model | | | | Multivariate model | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ |
| 20 | 50 | 3371 | 2484 | 1622 | 3 | 2627 | 2084 | 1588 | 85 |
| 40 | 70 | 4712 | 4624 | 4475 | 655 | 4170 | 4147 | 4112 | 2932 |
| 50 | 80 | 4822 | 4804 | 4740 | 1047 | 4363 | 4411 | 4420 | 3604 |
| 70 | 100 | 4896 | 4907 | 4895 | 1598 | 4515 | 4609 | 4680 | 4252 |
| 91 | 121 | 4897 | 4905 | 4897 | 1610 | 4484 | 4594 | 4674 | 4261 |

Table 4.4: Counts of scenarios where death benefit determined by corridor factors, Type A

Secondly, the counts of scenarios where the death benefit hits the corridor increase as the insured ages. They are not strictly increasing after the 70th policy year because of the increasing probability of policy lapse, but the trend exists. This phenomenon could be explained by the product features of the Type A contract and the rationale of corridor requirements. Per Section 2.2, the corridor factors constrain the risk sharing between the

insurer and the policyholder by setting the minimum death benefit for which the insurer must bear the risk in dollar amount as a multiple of $(c_y - 1)$ of account value, where $c_y$ denotes the corridor factor at attained age $y$ and $c_y$ decreases as the insured gets old. In the early years of the policy, despite the large value of $c_y$ compared to that applied when the insured is old, the account values are relatively small so that $c_y$-fold the account value could not reach the predetermined face amount, or the death benefit, for a Type A contract. Hence the death benefit remains the face amount. But as the account value grows, $c_y$ times the account value eventually exceeds the face amount, which is highly likely with the current premium payment plan and not offset by the declining multiplier $c_y$. Then the death benefit rises to the amount set by the corridor factors. Therefore, we see the upward trend in the counts over time in Table 4.4.

### Type B policy

Table 4.5 presents an opposite trend. Unlike a Type A contract where the increasing account value triggers an increase in the death benefit, the growth of the fund frees the death benefit from corridor requirements for a Type B product. As mentioned earlier, it is required that the ratio of the death benefits supported by the insurer and the policyholder be $(c_y - 1) : 1$, which is later referred to as the required ratio. The declining $c_y$ allows the proportion underwritten by the insurance company to shrink such that the mortality risk is gradually transferred to the policyholder. In other words, the required ratio decreases over time. On the other hand, the "contractual" ratio of NAAR to AV equals to $FA/AV$. With the face amount predetermined and fixed, an increasing account value makes the "contractual" ratio decrease, which agrees with the pattern of the required ratio. Considering the magnitude of the face amount and the projected account values, the "contractual" ratio starts at a higher point than the required ratio and then falls behind. For example, in the 70th policy year, the corridor factor drops to 1, and thus the required ratio becomes 0. No matter how large the account value grows to, $FA/AV$ could merely converge to 0 but never get to 0. This gap indicates that the insurance company has borne more than required. Therefore, the death benefit could be as much as the contract defines.

| Policy | Attained | Univariate model | | | | Multivariate model | | | |
|---|---|---|---|---|---|---|---|---|---|
| year | Age | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ | $w = 0$ | $w = 0.4$ | $w = 0.6$ | $w = 1$ |
| 20 | 50 | 5000 | 5000 | 5000 | 5000 | 4999 | 5000 | 5000 | 5000 |
| 40 | 70 | 4860 | 4873 | 4867 | 2566 | 4508 | 4617 | 4710 | 4677 |
| 50 | 80 | 4123 | 3616 | 2999 | 59 | 3190 | 2702 | 2211 | 52 |
| 70 | 100 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 91 | 121 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4.5: Counts of scenarios where death benefit determined by corridor factors, Type B

# Chapter 5

# Conclusion

As universal life insurance products become increasingly complex through diversified investment accounts, policyholders are facing challenges in predicting their account values and death benefits, which ultimately complicates their insurance planning. In this project, we have established univariate and multivariate stochastic asset models based on historical data for a simplified investment portfolio made up of two most commonly offered accounts: a savings account which tracks the returns of US 3-month T-bills and an equity account which ties to the performance of the S&P 500 index. Numerical results show that no matter whether these two assets are treated as independent or interdependent, the projected annual returns in the long run are roughly consistent. Additionally, models under both approaches on average understate T-bill returns but generate reliable S&P 500 returns. Note that empirical data suggest that residuals should be modelled by a distribution with heavier tails than the normal distribution used in this project. Hence the variability of future rates of return is also underestimated.

Based on the stochastic asset models, the cash flows of universal life policies with common product features and varying investment strategies are projected. Comments on the performance of universal life policies are summarized as follows.

- Policy lapse
  Although returns of the S&P 500 index are more volatile, they do not necessarily increase the probability of lapse. Nevertheless a safer investment strategy is to hold a balanced asset mix.

- Account values
  The account values tend to grow over time, but it is more likely to see declines in Type B policies' account values in the later years if the fund is dumped into the savings account. The distributions of future account values are right-skewed, which means that the favorable extremes significantly outperform the average level but happen

infrequently. A risky option such as the equity account does not always lead to worse downside potential than the conservative savings account, but it results in a wider range of possible outcomes. Meanwhile, the difference across asset models stands out. The univariate model gives higher projected account values as long as the equity account is involved in the asset mix, while the opposite holds if the entire fund is invested in the savings account.

- Death benefit
  For Type A policies, the death benefit in the later years tends to be determined by corridor factors. By contrast, the impact of the corridor factors on Type B policies' death benefits is more likely to be observed in the early years; as the insured ages, the death benefit shifts to the amount specified by the contract.

One limitation of this project rests on simulation of the asset returns over the insurance period. Investigation is needed regarding the performance of the two asset models in long term projections. For example, the GARCH(1,1) model is believed to provide good short-term forecasts for the volatilities of the S&P 500 index, while in the long term its monotonic term structure does not agree with the real trend. To address this issue, we could improve the models by including additional terms, say a component model with leverage effects for the volatilities of the S&P 500 index. But a full understanding is required for successful implementation and rational interpretation, which could be a barrier for practitioners to apply sophisticated models. Alternatively, re-estimation could be conducted regularly to reflect the current circumstances, and the method proposed in this project still applies.

Recall that in this project we suppose a static premium payment pattern where the policyholder keeps depositing a constant amount from age 30 to 65. However, such assumption is not realistic for universal life policies because the policyholder also manages his or her wealth via premium payments. For instance, if the fund is earning a low return, the policyholder would use the account value rather than the external fund to pay all deductions. Further study could be done with a dynamic premium strategy where the policyholder reacts to ups and downs of the investment returns and account values. Besides, one could explore how values of the parameters in the asset models influence cash flow projection through sensitivity tests. Last but not least, the condition for policy lapse in this project is set as the account value falls below zero. In practice, a rider called "No Lapse Guarantee" is offered by most insurance companies. This guarantee allows the policy to remain in force for the guaranteed period even if the cash value drops to zero as long as certain minimum premium payments have been made for a given period. The projection of cash flows for policies with no lapse guarantee is another interesting extension of this project.

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# Appendix A

# CUSUM test for structural changes

The Cumulative Sum (CUSUM) test aims to detect chances of structural changes such as change in lag length and in values of coefficients. The graphical outputs of the test are shown in Figure A.1. If there is no structural change, the empirical fluctuation process

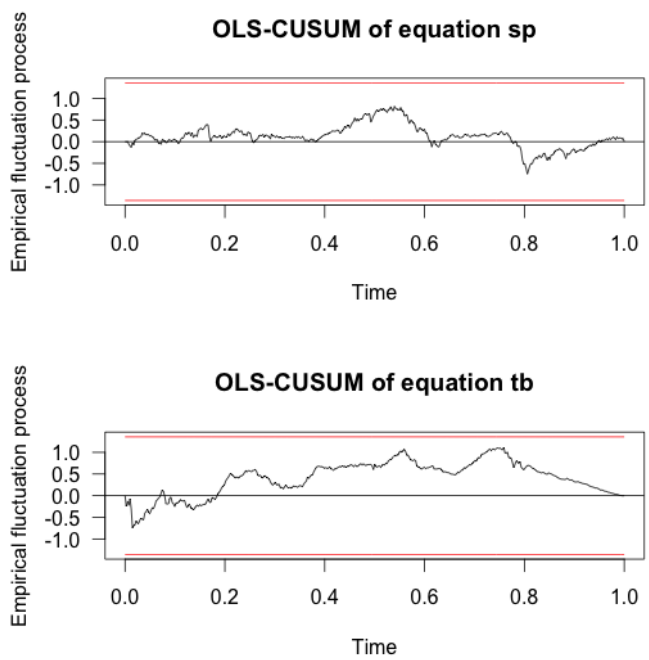


Figure A.1: OLS-CUSUM test on structural change, VAR(2)

computed based on the cumulative sum of the ordinary least square (OLS) residuals of the VAR model should not deviate from mean zero too much. As the asymptotic distribution of the fluctuation process is known, the thresholds of fluctuation, being crossed with a probability equal to the significance level, are computed and represented by red lines in Figure A.1. If the empirical fluctuation process shows large deviation and crosses the boundaries or thresholds, there is evidence that structural change should be expected. From Figure A.1, we see that the fluctuation process of $\{R_t\}$ is relatively stable. Despite the notable

deviation of the fluctuation process of $\{r_t\}$, it remains within the boundaries. In this sense, we conclude that structural changes are barely feasible.

# Appendix B

# Corridor factors

| Age | Factor | Age | Factor | Age | Factor |
|---|---|---|---|---|---|
| $\leqslant 40$ | 2.50 | 60 | 1.30 | 80 | 1.05 |
| 41 | 2.43 | 61 | 1.28 | 81 | 1.05 |
| 42 | 2.36 | 62 | 1.26 | 82 | 1.05 |
| 43 | 2.29 | 63 | 1.24 | 83 | 1.05 |
| 44 | 2.22 | 64 | 1.22 | 84 | 1.05 |
| 45 | 2.15 | 65 | 1.20 | 85 | 1.05 |
| 46 | 2.09 | 66 | 1.19 | 86 | 1.05 |
| 47 | 2.03 | 67 | 1.18 | 87 | 1.05 |
| 48 | 1.97 | 68 | 1.17 | 88 | 1.05 |
| 49 | 1.91 | 69 | 1.16 | 89 | 1.05 |
| 50 | 1.85 | 70 | 1.15 | 90 | 1.05 |
| 51 | 1.78 | 71 | 1.13 | 91 | 1.04 |
| 52 | 1.71 | 72 | 1.11 | 92 | 1.03 |
| 53 | 1.64 | 73 | 1.09 | 93 | 1.02 |
| 54 | 1.57 | 74 | 1.07 | 94 | 1.01 |
| 55 | 1.50 | 75 | 1.05 | $\geqslant 95$ | 1.00 |
| 56 | 1.46 | 76 | 1.05 | | |
| 57 | 1.42 | 77 | 1.05 | | |
| 58 | 1.38 | 78 | 1.05 | | |
| 59 | 1.34 | 79 | 1.05 | | |