Pricing Defaultable Catastrophe Bonds with Compound Doubly Stochastic Poisson Losses and Liquidity Risk

by

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Abstract

Catastrophe bond (CAT bond) is one of the modern financial instruments to transfer the risk of natural disasters to capital markets. In this project, we provide a structure of payoffs for a zero-coupon CAT bond in which the premature default of the issuer is also considered. The defaultable CAT bond price is computed by Monte Carlo simulations under the Vasicek interest rate model with losses generated from a compound doubly stochastic Poisson process. In the underlying Poisson process, the intensity of occurrence is assumed to follow a geometric Brownian motion. Moreover, the issuer's daily total asset value is modelled by the approach proposed in Duan et al. (1995), and the liquidity process is incorporated to capture the additional return of investors. Finally, a sensitivity analysis is carried out to explore the effects of key parameters on the CAT bond price.

Keywords: Catastrophe bond; premature default; stochastic interest rates; doubly stochastic Poisson process; liquidity process

Dedication

To my parents!

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Chapter 1

Introduction

1.1 Overview of catastrophe bonds

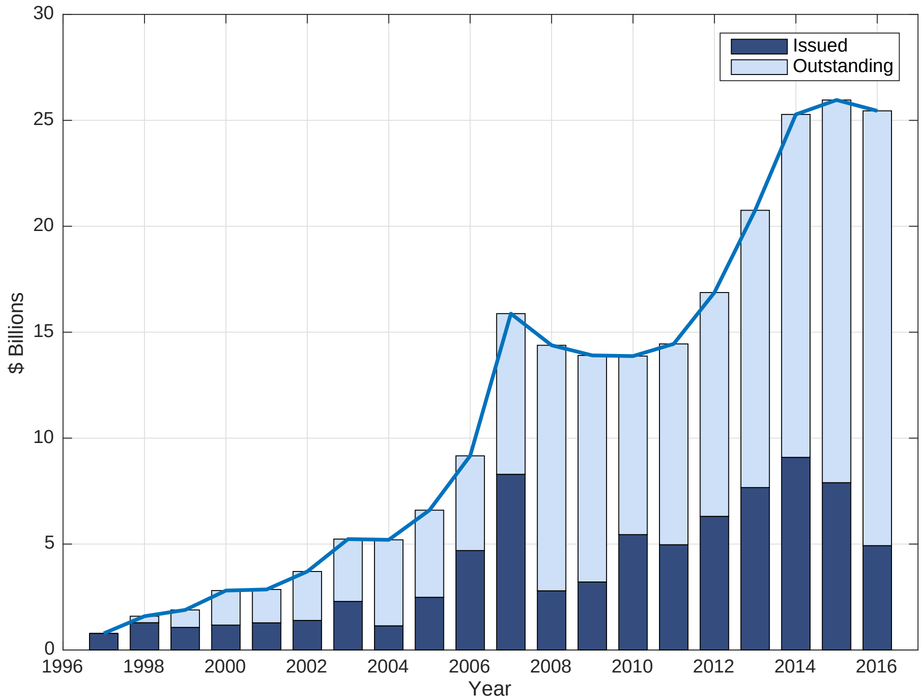
Natural disasters such us floods, earthquakes, or hurricanes can cause substantial financial losses with high uncertainty of occurrences, which is known as catastrophe risk. In other words, the risk of catastrophic events of high severity but low frequency may result in insurance companies to have insufficient reserves to cover insurance claims. This leads insurers or reinsurers to look for new financial or insurance instruments to make sure that they hold enough capital in case of future catastrophic events. One of examples of reducing reserve requirements and coverage cost is to issue a catastrophe bond (hereinafter referred to as CAT bond).

CAT bonds are well-known insurance-linked securities (short for ILS) used by insurance companies to transfer the risk of natural disasters to capital markets. They were invented after Hurricane Andrew hit southern Florida in 1992 causing about $25 billion in damage in today's dollars ("The Insurance Industry Has Been Turned Upside Down by Catastrophe Bonds", The Wall Street Journal, 2016). This hurricane is the costliest and most destructive damage ever at the time. Because of the unpredictable nature of the catastrophe, CAT bonds typically provide relatively higher yields to attract investors, offering an alternative way for investors who are seeking out higher returns.

Figure 1.1 shows the outstanding and new issued CAT bond and ILS market volumes from 1997 to the third quarter of 2016, according to the Artemis Deal Directory, a leading data provider. The data for this chart includes not only non-life CAT bonds, but also life-related ILS as well as other private transactions tracked by the Artemis Deal Directory.

As we observe from Figure 1.1, the market size of CAT bond and ILS proceeds at a slow pace until the financial crisis in 2008, followed by a significant growth. Roughly $7.8 billion of CAT bond and ILS issuance is witnessed for the year 2015, the third highest full-year

Figure 1.1: CAT bond and ILS Risk Capital



Source: Artemis Deal Directory
http : //www.artemis.bm/deal_directory/cat_bonds_ils_issued_outstanding.html

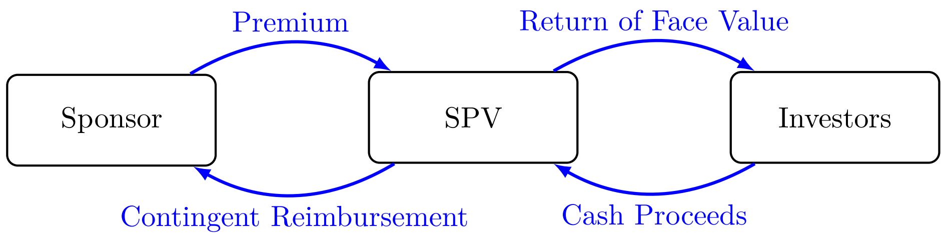
record ever. Additionally, according to the data from the Artemis Deal Directory, a new record for the total CAT bond issuance reaches $2.215 billion during the first quarter of 2016. Such an issuance volume represents about 31% growth over the previous record set in the first quarter of 2015. It is the first time in the history that the outstanding CAT bond limit was pushed to more than $26 billion at the end of a quarter. All these statistics showing an increasing volume of transactions in the CAT bonds market is anticipated.

1.2 Structure of a CAT bond transaction

As presented in Figure 1.2, the basic structure of CAT bonds includes three parties, a sponsor (insurer or reinsurer), a special purpose vehicle (short for SPV), and bond investors.

The sponsor enters into a risk transfer contract by paying premiums to a SPV, which is created specially for the transaction, in exchange for the coverage provided by the SPV via the issued securities. The SPV issues CAT bonds to investors (CAT bonds can also be issued by a traditional insurer) in the capital markets. In such a way the risk of disasters

Figure 1.2: Basic CAT bond structure



Source: Risk Management Solutions, Inc. (2012)

can be mitigated by shifting the risk to bond investors. The proceeds from the sale of bonds are then deposited in a collateral account managed by the SPV.

During the risk period, if specified events set in the CAT bond provisions do not occur, the bond investors are paid a competitive yield. As mentioned earlier, due to the uncertainty arising from rare events with dramatic consequences, these investors are compensated by higher interest rates in return for taking on the risk. The collateral account is then liquidated upon the expiration of CAT bonds and the investors are repaid with the full face value.

On the other hand, if a contingent event occurs and the accumulated loss amount reaches the trigger level, the investors sacrifice fully or proportionally their interest and face value. Proceeds will be withdrawn from the collateral account and reimburse the sponsor in order to help the sponsor pay claims arising from the events. From the perspective of the investors, the face value is repaid at risk because they could lose the entire face value in the SPV if the covered event is severe enough.

1.3 Trigger types

In the CAT bond market, three types of triggers that are commonly used are indemnity trigger, industry loss trigger, and parametric trigger. Table 1.1 summarizes that, for each type of trigger, what kind of coverage it provides. Generally, the use of a trigger involves a trade-off between moral hazard and basis risk (Doherty and Richter (2002)).

First, for a traditional indemnity trigger, the CAT bond is triggered by the actual claims or losses of the CAT bond's issuer. This type of trigger minimizes the issuer's basis risk, the gap between the bond payout and incurred losses. For this reason, this is the most advantageous for the issuer. However, it poses the danger of moral hazard that the issuer has an incentive to underwrite policies in a high catastrophe-risk area. The disadvantage

Table 1.1: Summary of three types of triggers

CAT bonds with	Coverage based on
indemnity trigger	issuer's actual excess claims
industry loss trigger	total industry losses on a catastrophic event
parametric trigger	exceedance of specific event parameters

of using the indemnity trigger is that it may require a longer waiting period until the settlement of all claims has been reached, so it tends to be infavouered by the investors.

Second, unlike an indemnity trigger that is designed such that the debt is forgiven in direct relationship to the issuer's actual claims or losses, an industry loss trigger is defined by the estimate of the total losses experienced by the industry after a certain catastrophic event. Since the industry loss trigger is determined by considering the experiences of many companies and is provided by an independent third-party service (e.g., Property Claims Services in the U.S. and PERILS in Europe), it is more transparent than the indemnity trigger and consequently lessens the moral hazard. In addition, industry loss triggers can be measured more quickly after a catastrophic event than traditional indemnity triggers. Therefore, industry loss triggers become more advantageous for investors. On the other hand, this type of trigger exposes the issuer to a higher level of basis risk because the amount of claims it has to pay may not be exactly the same as the issuer's share of the industry loss.

Third, the parametric trigger is based on the occurrence of a natural disaster, such as an earthquake exceeding magnitude of 7.0. Apparently, it is more understandable for the investors than the probability of an issuer incurring a certain amount of losses. According to Cummins (2008), a parametric trigger subjects the issuer to the highest degree of basis risk. If the location or geographical area where the loss is measured is defined appropriately, this type of parameters can roughly predict the issuer's size of claims, and results in a lower degree of basis risk. For CAT bonds with a parametric trigger, they are very transparent, and the moral hazard risk is the lowest among these three types of triggers because it is difficult for an issuer to manipulate losses. Moreover, the claims are settled even more quickly because the parameter of an event is available shortly after a triggering event. As a result, it is the most attractive to the investors.

Overall, these triggers either reduce the sponsor's moral hazard behaviour but create the basis risk or the other way around, which supports the evidence that a trade-off exists between moral hazard and basis risk. According to the statistics available on Artemis, most CAT bonds use either an indemnity trigger or an industry loss trigger. In this project, we focus on the CAT bond contracts based on indemnity triggers, where the payouts are directly tied to the losses of a CAT bond's issuer, due to its popularity.

1.4 Motivation

Lee and Yu (2002) evaluated the zero-coupon defaultable CAT bonds for a log-normal loss process by applying Monte Carlo simulations. However, there exists the possibility that a CAT bond's issuer goes bankrupt before the bond matures. This project differs from Lee and Yu (2002) by considering the early default arrival as shown in Black and Cox (1976).

As pointed out in Lin et al. (2009), a pure Poisson process is not appropriate to describe the arrival process of future natural disasters because of the global climate change in recent decades. An increasing exponential trend of the annual number of natural catastrophic events can be observed in the U.S. over 54 years starting 1950, which supports the adoption of a compound Poisson process with stochastic intensity. Based on their findings, when pricing a CAT bond, we take into account the assumption that the dynamic aggregate losses of an insurance company follow a compound doubly stochastic Poisson process with lognormal distribution of the loss magnitudes to represent this phenomenon.

Besides, there is no literature regarding pricing of a CAT bond with liquidity risk involved. Liquidity refers to an investor's ability to quickly sell or purchase a security at the price that reflects the current market conditions without causing a drastic change in the price. An important characteristic of a liquid market is that there are always willing buyers and sellers who are ready to enter transactions. Conversely, in an illiquid market, it is often characterized by a big trade-off between how soon and how much the security can be sold for. In such a market, liquidity risk influences all market participants. This project completes the literature by incorporating the liquidity process (see Longstaff et al. (2005)) to capture the extra return that investors may require. This extra return can be regarded as the compensation for risks borne by the bond investors, differing from the investors who hold riskless securities.

1.5 Outlines

The rest of this project is organized as follows. The following chapter gives an overview of articles in this area.

In Chapter 3, we first introduce notations and assumptions that are used in this project. Then, the models for CAT bond issuers, including dynamics of interest rate, asset, loss, liquidity and the frequency of catastrophic events, are presented. Furthermore, we specify the payoffs of a zero-coupon CAT bond under different scenarios. The pricing formulae are also provided.

Chapter 4 prices one-year zero-coupon CAT bonds through Monte Carlo simulations for illustrations. Sensitivity analysis is carried out as well to explore how the CAT bond price varies with key parameters. Finally, Chapter 5 concludes the project with some general remarks.

Chapter 2

Literature review

Pricing securities subject to default risk have been studied by a number of articles. For example, Merton (1974), Duffie et al. (1996), Lando (1998), Hui et al. (2003), and Longstaff et al. (2005). Merton (1974) presented a risk structure of interest rates with the probability of default. Instead of using the classical approach in Merton (1974), Duffie et al. (1996) proposed an intensity-based framework by deriving a risk-neutral valuation formula to price defaultable securities, in which the time of default with arrival intensity itself is a random process. Lando (1998) also presented a similar framework, where the default intensity is assumed to follow a doubly stochastic Poisson process, also known as the Cox process. Based on the result from Duffie et al. (1996), Duffie et al. (1997) proposed an alternative way of valuing swaps by discounting the future cash flows with a default and liquidity-adjusted instantaneous short rate. Longstaff et al. (2005) pointed out that most corporate bond spreads are caused by default risk, and incorporated interest rate and liquidity processes, apart from the default process, to develop a closed-form expression for pricing a corporate bond. Each of the processes is stochastic and independent of each other. However, few researchers attempted to price defaultable CAT bonds.

Another important issue to be addressed when pricing CAT bonds is the possibility that a CAT bond's issuer goes bankrupt and becomes insolvent before the CAT bond expires. In the model of Merton (1974), default refers to the event when the firm's value at maturity falls below its liability. However, the underlying assumption is that the default event may only occur at the maturity date of a bond. Later, Black and Cox (1976) relaxed the assumption of Merton (1974) by taking into account the premature default. The concept of safety covenants was applied in the model of Black and Cox (1976). That is, bond holders have the right to force the firm to go bankrupt or reorganize if the value of the firm's assets is lower than a certain barrier. Briys and De Varenne (1997) proposed a corporate bond valuation model where both early default and interest rate risk are considered.

On the issuer's assets side, the dynamics of a firm's value is governed by a geometric Brownian motion in the framework of Merton (1974), with the assumption of a flat term structure. Subsequently, Duan et al. (1995) extended Merton's (1974) asset price dynamics by relating the asset model to the Vasicek's (1977) mean-reverting stochastic interest rate model to evaluate the interest rate risk exposures of banks. In order to analyze interest rate risk characteristics of corporate bonds, Nawalkha (1996) also generalized Merton's (1974) model to include Vasicek's (1977) term structure model. In this project, we use the same setting as Duan et al. (1995) and Nawalkha (1996) that incorporated Vasicek (1977) process into Merton's (1974) model for asset dynamics to obtain the daily asset value of a CAT bond's issuer.

As for the aggregate loss model, a typical way in risk theory is to use a compound Poisson process; see Bowers et al. (1986) for details. In a compound Poisson process, our interest centres on the sum of the loss random variables for all loss arrivals following a Poisson process up to a certain time. In the case of CAT bonds, these variables are positive random severities that are independent and identically distributed, and are independent of the underlying Poisson process for loss frequencies. Baryshnikov et al. (2011), Lee and Yu (2002), and Jaimungal and Wang (2006) modelled the frequency and severity of catastrophes with compound Poisson processes to evaluate the prices of different CAT insurance products. All of these articles used a constant arrival rate of catastrophic events within the process to describe dynamic losses. However, Jang (2000) indicated that the appropriateness of using a Poisson process as the loss arrival process in insurance modelling is questioned, and thus employed a doubly stochastic Poisson process to measure the number of losses due to catastrophic events, for deriving pricing formulae for a stop-loss reinsurance contract. In addition, Lin et al. (2009) indicated that historical data regarding adjusted number of natural catastrophic events shows an upward trend from 1950 to 2004 in the U.S., and suggested that a doubly stochastic Poisson process would be more adequate than using an average constant occurrence rate of catastrophes. To keep up with this trend, we assume a compound doubly stochastic Poisson process for the aggregate losses in this project.

This project follows the existing literature on the valuation of default-risky CAT bonds. An important theoretical work in this area includes Lee and Yu (2002), which assumed the interest rate dynamics of Cox et al. (1985) and a compound Poisson process for the aggregate loss dynamics. As previously mentioned, we adopt the one-factor interest rate model by Vasicek (1977), which has the disadvantage of producing negative values. However, the Central Banks of Denmark, Sweden, Switzerland and Japan recently decided to adopt negative interest rate policy. Thus, it becomes reasonable to model interest rate movements with the Vasicek (1977) process nowadays.

The objective of this project is to incorporate the interest rate and liquidity risks, early default time, and compound doubly stochastic Poisson process into the pricing model of a zero-coupon CAT bond.

Chapter 3

The model

In this chapter, we present the stochastic interest rate model, the asset dynamics, the aggregate loss dynamics, and the liquidity process of a CAT bond's issuing firm, along with the model assumptions and notations. Throughout this project, we assume a CAT bond matures at time T; the subscript t ($t < T$) is measured in days, unless stated otherwise, and an issuer is the issuing firm of a CAT bond, either a traditional insurer or a special purpose vehicle (SPV). Furthermore, we provide the payoff structures for CAT bonds without and with default risk, respectively, as well as their pricing formulae.

3.1 Issuers of CAT bonds

3.1.1 The stochastic interest rate model

Interest rate models have been used widely in the valuation of interest rate derivatives. Two of one-factor interest rate models that are widely used in the finance literature are the Vasicek (1977) model and the Cox, Ingersoll and Ross (1985) model. The former allows negative interest rates while the latter ensures that interest rates are always non-negative. Both of them are driven by only one source of market risk. Furthermore, they both have mean-reverting property that ensures a trajectory of interest rate will revert toward the long-term mean level in the long run.

Jaimungal and Wang (2006) and Lin et al. (2009) proposed pricing formulae for catastrophe put options with the Vasicek model. Lin et al. (2009) relaxed the assumption in Jaimungal et al. (2006) to include a stochastic catastrophic process. In addition, Nowak et al. (2012) priced zero-coupon CAT bonds with the Vasicek interest rate model. We follow their setting and assume the interest rates are governed by the Vasicek model.

Let r_t be the interest rate. Assuming that r_t follows the Vasicek (1977) model with parameters κ, θ, and σ_r, the stochastic differential equation is given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_{r,t}, \tag{3.1}$$

where κ is the mean-reverting force measuring the effect of pulling the process back to its long-term mean θ; σ_r is the volatility of the interest rate; and $W_{r,t}$ is a Weiner process modelling the randomness of market risk.

3.1.2 The asset dynamics

In Merton (1974), the dynamics for the firm's assets is modelled by a lognormal diffusion process. However, Lee and Yu (2002) pointed out that this typical approach of modelling asset value fails to capture the impact of stochastic interest rates, since the issuers invest most of their proceeds from the sale of CAT bonds in treasury securities or other interest-sensitive assets.

For this reason, we adopt the asset model which depicts the asset value consisting of interest rate risk and credit risk, where credit risk is independent of interest rate risk. More details can be found in Duan et al. (1995). Specifically, the asset dynamics is written as

$$\frac{dV_t}{V_t} = r_t dt + \phi \sigma_r dW_{r,t} + \sigma_V dW_{V,t}, \tag{3.2}$$

where V_t represents an issuer's asset value at time t, $t \in [0, T]$; r_t is the interest rate at time t; ϕ can be interpreted as the interest rate elasticity of an issuer's total assets; σ_r and σ_V are the volatilities of the interest rate risk and credit risk, respectively; and $W_{V,t}$ denotes a Weiner process independent of $W_{r,t}$.

3.1.3 The aggregate loss dynamics

A classic model for the distribution of the aggregate loss over a time period in actuarial literature is the compound Poisson process; see, for example, Bowers et al. (1986). Yet an increasing trend in the number of future catastrophic events is anticipated due to global warming, according to Lin et al. (2009). Therefore, as opposed to using a deterministic intensity rate, we assume a stochastic intensity rate to model the arrival of catastrophic events. The aggregate loss model is then known as a compound doubly stochastic Poisson process. It provides more flexibility to model the rate of occurrence of catastrophic events because the rate itself is stochastic and time-dependent. The following introduces the models for loss frequency and severity of catastrophic events, and their assumptions.

1. **Arrival process for catastrophic events**: define λ_t^C as the stochastic intensity rate at time t that follows a geometric Brownian motion. The process can be expressed as

$$\frac{d\lambda_t^C}{\lambda_t^C} = \mu_\lambda dt + \sigma_\lambda dW_{\lambda,t}, \tag{3.3}$$

where μ_λ and σ_λ are the instantaneous change rate and the volatility of change rate of the catastrophic intensity, respectively, and $W_{\lambda,t}$ is a Weiner process. Note that the superscript C is short for the term "catastrophic events".

Given an initial value λ_0^C, the stochastic differential equation (3.3), by *Itô*'s lemma, has an analytical solution given by

$$\lambda_t^C = \lambda_0^C \times \exp\left(\mu_\lambda - \frac{1}{2}\sigma_\lambda^2 + \sigma_\lambda W_{\lambda,t}\right), t \geq 0. \tag{3.4}$$

2. **Counting process**: denote $N(t)$ as the number of catastrophic events up to time t with $N(0) = 0$, and assume it follows a Poisson process parameterized by the stochastic rate λ_t^C.

3. **Loss severity**: let X_j $(j \geq 1)$ represent the loss amount incurred from the j^{th} catastrophe. We assume that X_j's are independent, identical, and log-normally distributed random variables with parameters μ_C and σ_C. We also assume that all severities X_j's are independent of the counting process $N(t)$.

4. **Aggregate loss process**: based on the assumptions above, the aggregate loss distribution is then known as a compound doubly stochastic Poisson process. Let C_t be the aggregate loss of an issuer at time t given by

$$C_t = \sum_{j=1}^{N(t)} X_j$$

with $C_t = 0$ if $N(t) = 0$.

3.1.4 The liquidity process

Liquidity risk is a financial risk that an investor cannot meet the short term financial demands. An illiquidity market has characteristics that both the trading volume and frequency are low. Typically, financial instruments with lower trading volume tend to expose the investors to a higher degree of liquidity risk since the market demand cannot easily match with the market supply condition. Because of the characteristic, investors may require an extra yield on the assets in return for their inability to convert the assets to cash quickly. This extra return can be treated as an incentive for investors in the capital markets to invest in such risky securities.

Following Longstaff et al. (2005), the liquidity dynamics γ_t is assumed to be governed by the following stochastic process

$$d\gamma_t = \sigma_\gamma dW_{\gamma,t}, \tag{3.5}$$

where σ_γ is a positive constant and $W_{\gamma,t}$ is a Weiner process.

3.1.5 Other assumptions and notations

Some other notations and assumptions regarding the issuer of a CAT bond are given as follows.

1. **Independence between processes**: Longstaff et al. (2005) assumed the dynamics of default intensity, interest rate, and liquidity are independent of each other to simplify the model. Following this idea, for simplicity, we make the same assumption that each of the Weiner processes in r_t, V_t, λ_t^C and γ_t is independent of each other.

2. **Face value**: let L be the face value of an issuer's total debts at maturity.

3. **Bankruptcy level**: we let K_t^D, similarly as in Bielecki and Rutkowski (2013), denote the threshold of a poor performance of the issuer's assets set by Black and Cox (1976), in terms of a time-dependent deterministic barrier, given by

$$K_t^D = \begin{cases} K \cdot e^{-\gamma(T-t)} & \text{for} \quad t \in [0,T) \\ L & \text{for} \quad t = T \end{cases},$$

   for some constant $K > 0$. It is also known as a safety covenant, which provides bond holders the right to take over a firm if the value of that firm's assets after a certain catastrophic event is less than the contractual threshold K_t^D. Note that the superscript D is short for the term "default" hereafter.

   In this project, it is assumed that γ equals 0. The bankruptcy level can then be simplified to K^D, a constant over time.

4. **Default time**: let τ_d be the first time when an issuer's total assets V_t after deducting the aggregate loss C_t falls below the level K^D. This includes default prior to or at the maturity date. The default time equals ($\inf \emptyset = +\infty$)

$$\tau_d = \inf\{t \in [0,T] : V_t - C_t < K^D\}.$$

3.2 Zero-coupon CAT bonds

As mentioned earlier, we study only the indemnity loss trigger in this project. The following specifies the assumptions of a zero-coupon CAT bond we use in this project, including the trigger barrier, the trigger time, and the payoffs.

1. **Recovery rate**: we first define $\beta \in [0,1]$ as the recovery rate that determines the recovery payoff received at default time, if default occurs prior to or at the CAT bond's maturity date.

2. **Trigger level**: denote K_j^C as the annual indemnity loss trigger level set in the CAT bond provisions. According to the property for a compound Poisson process that the value of each arrival is independent of the underlying Poisson process, K_j^C can be determined by
$$K_j^C = \lambda_j^C \times \exp\left(\mu_C + \frac{\sigma_C^2}{2}\right),$$
which is the expectation of the compound aggregate loss process, C_t, based on the Poisson process $N(t)$ with intensity rate λ_j^C and log-normal severity with mean μ_C and variance σ_C^2. Here, λ_j^C represents the rate of occurrence for the j^{th} year and is assumed constant over the entire j^{th} year, $j = 1, 2, ..., T$. Therefore, K_j^C is a function that is piecewise constant.

3. **Trigger time**: in the case when an indemnity trigger is being used, we denote τ_c as the first time $t \in [0,T]$ when the value of the aggregate loss (C_t) of an issuer exceeds K_j^C. Specifically,
$$\tau_c = \inf\{t \in [0,T] : C_t > K_j^C;\ j-1 < t \leq j,\ j = 1, 2, ..., T\}.$$
It is assumed that the CAT bond is triggered only once by or at the maturity date T.

4. **Payoffs**: denote the payoff by $PO_i(time)$, where the subscript i changes according to which case is being discussed, and $time$ indicates when the payment is made. For example, $i = f$ and $i = d$ mean the cases of default-free and default-risky CAT bonds, respectively.

   (a) We begin with the simplest case where there are no default risk involved in pricing a CAT bond. In this special case, the future bond payoffs are paid out to the bond holders only when the CAT bond expires. Let $PO_f(T)$ be the payoff at time T; the payoff of a default-free CAT bond at maturity can be written as

$$PO_f(T) = \begin{cases} L & \text{if} \quad C_T \leq K_j^C, \\ a \cdot L & \text{if} \quad C_T > K_j^C. \end{cases} \tag{3.6}$$

If $C_T \leq K_j^C$, then the bond holders receive whole face value L at maturity. Otherwise the bond holders receive only a partial face value $a \cdot L$, where $a \in [0, 1)$ is a pre-specified proportion of the face value needed to be paid to the bond holders when the established trigger has been reached.

(b) Now we consider a CAT bond with default risk; the payoff is determined according to the time of default occurring (default after, at, or by the maturity date) and are specified below.

i. The event of default does not occur during the risk period of the CAT bond (i.e., $\tau_d > T$), so the payment is made at time T. That is

$$PO_d(T) = \begin{cases} L & \text{if} \quad C_T \leq K_j^C \text{ and } V_T - C_T \geq L, \\ a \cdot L & \text{if} \quad C_T > K_j^C \text{ and } V_T - C_T \geq a \cdot L. \end{cases} \tag{3.7}$$

This payoff structure is an extension of (3.6), with the consideration of default risk. Similar concept as in (3.6), for the case when the trigger has not been pulled, the bond holders receive L at maturity; whereas if the CAT bond is triggered, they lose a portion of the face value and receive $a \cdot L$.

ii. The issuer is insolvent and defaults at the maturity date of the CAT bond, i.e. $\tau_d = T$. Then

$$PO_d(T)$$
$$= \begin{cases} \max\{\beta \cdot (V_T - C_T),\ 0\} & \text{if} \quad C_T \leq K_j^C \text{ and } V_T - C_T < L, \\ \min\{a \cdot L,\ \max[\beta \cdot (V_T - C_T),\ 0]\} & \text{if} \quad C_T > K_j^C \text{ and } V_T - C_T < a \cdot L, \end{cases} \tag{3.8}$$

where $0 \leq \beta < 1$. The upper term in (3.8) happens when the underlying aggregate loss does not exceed the trigger level K_j^C but the issuer defaults at the bond's maturity date. In this case, the bond holders receive either a portion of the issuer's remaining asset value or nothing at time T. Whereas the lower term in (3.8) represents the scenario that the CAT bond is triggered and the issuer is insolvent. In this case, the bond holders get the minimum of $a \cdot L$ and $\max[\beta \cdot (V_T - C_T),\ 0]$ when the CAT bond matures.

iii. The event of default occurs before the CAT bond expires (i.e., premature default, $\tau_d < T$), so the payment is made at τ_d and can be written as

$$PO_d(\tau_d) = \begin{cases} \beta \cdot K^D & \text{if} \quad \tau_c > \tau_d, \\ \min\{a \cdot L \cdot e^{-\int_{\tau_d}^{T} (r_t + \gamma_t)\, dt},\ \beta \cdot K^D\} & \text{if} \quad \tau_c \leq \tau_d. \end{cases} \tag{3.9}$$

The upper term in (3.9) includes two scenarios, for which the investors of the CAT bond receive $\beta \cdot K^D$. One is that the trigger is not pulled by or at time T ($\tau_d < T < \tau_c$), and the other is that the trigger is pulled after the issuer defaults and occurs by or at time T ($\tau_d < \tau_c \leq T$).

The lower term in (3.9) shows the scenario that the issuer defaults after the CAT bond is triggered ($\tau_c \leq \tau_d$). The payoff $a \cdot L$ paid at time T includes the discount factor "$\exp\left[-\int_{\tau_d}^{T} (r_t + \gamma_t)\, dt)\right]$" because it is compared to the payoff $\beta \cdot K^D$ when the issuer defaults on its debts at time τ_d. The issuer pays out the minimum of the two quantities at τ_d, the default time of the CAT bond's issuer.

Table 3.1 summarizes all possible payoffs stated in (3.7)− (3.9) for the CAT bond forgiven on the issuer's actual losses. Scenarios i., ii., and iii. describe the events of default that does not occur during the CAT bond term, occurs at maturity, and occurs prior to time T, respectively.

It is worth noticing that the payoffs are made at time T for both i. and ii., whereas for iii. the investors receive the payoff at time τ_d if the issuer defaults before the CAT bond's maturity date. As might be expected, the events for all scenarios in Table 3.1 are disjoint. In order for us to investigate the scenario-specific effect on the total price of a CAT bond, we specify each row in the condition column of Table 3.1 as Scenarios 1, 2, 3, 4, 5-1, 5-2 and 6, starting from top to bottom.

5. **Price at time 0**: once the payoffs of a CAT bond are known, we can then price the CAT bond at its issuing date (i.e., time 0). Denote $P_i(0, T)$ as the price at time 0 discounting from time T, and i as discussed in Section 3.2. According to the specified interest rate, asset, aggregate loss, and liquidity dynamics, the CAT bond's pricing

Table 3.1: Summary of payoffs under different scenarios

	Issuer defaults or not	CAT bond is triggered or not	Condition	Payoff (at T for i. and ii.; and at τ_d for iii.)
i.	No.	No.	$\{\tau_d > T, \tau_c > T, V_T - C_T \geq L\}$	L
	No.	Yes.	$\{\tau_d > T, \tau_c \leq T, V_T - C_T \geq a \cdot L\}$	$a \cdot L$
ii.	Yes, at time T.	No.	$\{\tau_d = T, \tau_c > T, V_T - C_T < L\}$	$\max\{\beta \cdot (V_T - C_T),\ 0\}$
	Yes, at time T.	Yes.	$\{\tau_d = T, \tau_c \leq T, V_T - C_T < a \cdot L\}$	$\min\{a \cdot L,\ \max[\beta \cdot (V_T - C_T),\ 0]\}$
iii.	Yes, before time T.	No.	$\{\tau_d < T < \tau_c\}$	$\beta \cdot K^D$
		Yes, after the issuer defaults.	$\{\tau_d < \tau_c \leq T\}$	
	The issuer defaults by time T and after the CAT bond is triggered.		$\{\tau_c \leq \tau_d < T\}$	$\min\{a \cdot L \cdot e^{-\int_{\tau_d}^{T} (r_t + \gamma_t)\, dt},\ \beta \cdot K^D\}$

formulae are provided below.

(a) Based on the payoff structure in (3.6), the CAT bond can be valued by the discounted expectation of the payoffs of two disjoint events, $\tau_c > T$ and $\tau_c \leq T$. The pricing formula is given by

$$\begin{aligned} P_f(0,T) =& E\left[PO_f(T) \cdot e^{-\int_0^T (r_t + \gamma_t)\, dt}\right] \\ =& E\left[L \cdot e^{-\int_0^T (r_t + \gamma_t)\, dt} \mathbb{1}_{\{\tau_c > T\}}\right] + E\left[a \cdot L \cdot e^{-\int_0^T (r_t + \gamma_t)\, dt} \mathbb{1}_{\{\tau_c \leq T\}}\right] \\ =& E\left[L \cdot e^{-\int_0^T (r_t + \gamma_t)\, dt} \mid \tau_c > T\right] \times Pr(\tau_c > T) \\ &+ E\left[a \cdot L \cdot e^{-\int_0^T (r_t + \gamma_t)\, dt} \mid \tau_c \leq T\right] \times Pr(\tau_c \leq T). \end{aligned} \tag{3.10}$$

This pricing formula involves interest rate and liquidity risks; the randomness of catastrophe intensity is also embedded, where $\mathbb{1}$ is an indicator function. We observe that the payoffs in each of the terms are discounted at the adjusted rate $r_t + \gamma_t$.

In the terms on the right-hand side of the second equality in (3.10), the former is the expected present value of the promised payment to the investors when the CAT bond is not triggered during the risk period of the CAT bond (i.e., $\tau_c > T$), and the latter is the expected present value of the payment when a specified event occurs that meets the trigger condition to activate a payout (i.e., $\tau_c \leq T$). Obviously, the intersection of these two events $\{\tau_c > T\}$ and $\{\tau_c \leq T\}$ is empty. Next, the terms on the right-hand side of the last equality in (3.10) are obtained by applying the law of total expectation.

(b) Based on the payoff structure in (3.7)$-$(3.9), and the property that the events for all scenarios are mutually exclusive and exhaustive, the price of the CAT bond at time 0 can be written as

$$
\begin{aligned}
&P_d(0,T)\\
&= E\left[L \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mathbb{1}_{\{\tau_d>T, \tau_c>T, V_T-C_T\geq L\}}\right]\\
&\quad + E\left[a \cdot L \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mathbb{1}_{\{\tau_d>T, \tau_c\leq T, V_T-C_T\geq a\cdot L\}}\right]\\
&\quad + E\left[\max\{\beta \cdot (V_T - C_T),\ 0\} \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mathbb{1}_{\{\tau_d=T, \tau_c>T, V_T-C_T<L\}}\right]\\
&\quad + E\left[\min\{a \cdot L,\ \max[\beta \cdot (V_T - C_T),\ 0]\} \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mathbb{1}_{\{\tau_d=T, \tau_c\leq T, V_T-C_T<a\cdot L\}}\right]\\
&\quad + E\left[\beta \cdot K^D \cdot e^{-\int_0^{\tau_d} (r_t+\gamma_t)\,dt} \mathbb{1}_{\{\tau_d<T<\tau_c \text{ or } \tau_d<\tau_c\leq T\}}\right]\\
&\quad + E\left[\min\{a \cdot L \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt},\ \beta \cdot K^D \cdot e^{-\int_0^{\tau_d} (r_t+\gamma_t)\,dt}\} \mathbb{1}_{\{\tau_c\leq\tau_d<T\}}\right].
\end{aligned}
\tag{3.11}
$$

This pricing formula differs from (3.10) by taking into account the default risk and the event of premature default. The first two terms in the expression represent the expected present values of future payoffs when there is no event of default occurs during the CAT bond term. The third and forth terms show the expected present values of payoffs paid at time T when the issuer defaults at the maturity date, no matter whether the CAT bond has been triggered or not. Finally, the last two terms are the expected present values of future payoffs in the event of an premature default. Since all the conditions stated in each of the expressions above are mutually exclusive and exhaustive, (3.11) can be rewritten as the sum of all the products of a conditional mean and corresponding probability by applying the law of total expectation. More specifically,

$$
\begin{aligned}
&P_d(0,T) \\
&= E\left[L \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mid \tau_d > T, \tau_c > T, V_T - C_T \geq L\right] \\
&\qquad \times Pr(\tau_d > T, \tau_c > T, V_T - C_T \geq L) \\
&\quad + E\left[a \cdot L \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mid \tau_d > T, \tau_c \leq T, V_T - C_T \geq a \cdot L\right] \\
&\qquad \times Pr(\tau_d > T, \tau_c \leq T, V_T - C_T \geq a \cdot L) \\
&\quad + E\left[\max\{\beta \cdot (V_T - C_T),\ 0\} \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mid \tau_d = T, \tau_c > T, V_T - C_T < L\right] \\
&\qquad \times Pr(\tau_d = T, \tau_c > T, V_T - C_T < L) \\
&\quad + E\left[\min\{a \cdot L,\ \max[\beta \cdot (V_T - C_T),\ 0]\} \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt} \mid \tau_d = T, \tau_c \leq T, V_T - C_T < a \cdot L\right] \\
&\qquad \times Pr(\tau_d = T, \tau_c \leq T, V_T - C_T < a \cdot L) \\
&\quad + E\left[\beta \cdot K^D \cdot e^{-\int_0^{\tau_d} (r_t+\gamma_t)\,dt} \mid \tau_d < T < \tau_c \text{ or } \tau_d < \tau_c \leq T\right] \\
&\qquad \times Pr(\tau_d < T < \tau_c \text{ or } \tau_d < \tau_c \leq T) \\
&\quad + E\left[\min\{a \cdot L \cdot e^{-\int_0^T (r_t+\gamma_t)\,dt},\ \beta \cdot K^D \cdot e^{-\int_0^{\tau_d} (r_t+\gamma_t)\,dt}\} \mid \tau_c \leq \tau_d < T\right] \\
&\qquad \times Pr(\tau_c \leq \tau_d < T).
\end{aligned}
\tag{3.12}
$$

Chapter 4

Numerical Results

This chapter calculates the price of a defaultable zero-coupon CAT bond using Monte Carlo simulations, based on the assumptions we made and the pricing formulae we derived in Chapter 3. The simulations are run with $100,000$ paths, on a daily basis. In addition, throughout this chapter, it is assumed that the CAT bond can only be triggered once during its risk period.

This chapter is organized as follows. Section 4.1 provides the methodology regarding how we simulate each of the stochastic processes and obtain the simulated price of the CAT bond. The steps taken in order to calculate the price of the defaultable zero-coupon CAT bond are also provided. In Section 4.2, the general assumptions used for numerical illustrations are given. Afterwards, we compute the price of the bond for a representative base case, including the bonds without and with the default risk. In the last section, we carry out a sensitive analysis to study how the bond price varies with key parameters.

4.1 Simulation methodology

From Sections 4.1.1 to 4.1.3, we briefly introduce how to simulate the stochastic processes that we discussed in Chapter 3. Unless the process itself has an analytic solution, instead of assuming that a process evolves continuously, a typical approach to modelling a continuous process is to partition the underlying interval $[0, T]$ into tiny subintervals of equal width Δt and calculate the value of the process at each of the discrete time points, $t_j = j \cdot \Delta t$, $j = 1, 2, \ldots, (T/\Delta t)$. By doing this, the computation process is simplified. With this in mind, we provide the approximation formulae for (3.10) and (3.12) in Section 4.1.4. Lastly, Section 4.1.5 describes the steps for simulation procedure.

4.1.1 Generate interest rates

We adopt the Vasicek (1977) process, a mean reverting short rate process, to model the interest rates. Its stochastic differential equation is given by (3.1). With a specific initial rate of return r_0, the trajectory of the short rate can be generated by the following equation (see Gillespie (1996) for more details):

$$r_t = e^{-\kappa\Delta t} \cdot r_0 + (1 - e^{-\kappa\Delta t}) \cdot \theta + \sigma_r \cdot \sqrt{\frac{1 - e^{-2\kappa\Delta t}}{2\kappa}} \cdot \sqrt{\Delta t} \cdot N_r(0,\, 1), \tag{4.1}$$

where

- $t = t_j = j \cdot \Delta t$ and $\Delta t = t_j - t_{j-1} = 1/365$, $j = 1, 2, \ldots, T/\Delta t$;
- the parameters κ, θ and σ_r are defined in Chapter 3; and
- $N_r(0,\, 1)$ denotes the random number generated from the standard normal distribution.

4.1.2 Generate issuer's total assets

The asset dynamics is presented in (3.2), the Merton's (1974) model with the consideration of the interest rate risk (see Duan et al. (1995)). Given the initial value V_0, the value of the issuer's total assets at time t by *Itô*'s lemma is given as

$$V_t = V_0 \times \exp\left[\int_0^t \left(r_s - \frac{\phi^2 \cdot \sigma_r^2 + \sigma_V^2}{2}\right) ds + \phi \cdot \sigma_r \cdot N_r(0, t) + \sigma_V \cdot N_V(0, t)\right], \tag{4.2}$$

where

- r_s is the interest rate at time s;
- the parameters ϕ, σ_r and σ_V are defined in Chapter 3; and
- $N_r(0,\, t)$ and $N_V(0,\, t)$ represent the generated normal random numbers with mean 0 and variance t for interest rate and asset value components, respectively.

Since both σ_V^2 and σ_r^2 are pre-specified values, and $\int_0^t r_s\, ds$ is approximated by $\sum_{i=1}^{j} r_{t_i} \times \Delta t$, (4.2) can be rearranged and rewritten as follows, in which the sample values of V_t are computed at the discrete time points $t = t_j = j \cdot \Delta t$ and $\Delta t = t_j - t_{j-1} = 1/365$, $j = 1, 2, \ldots, T/\Delta t$.

$$V_t \approx V_0 \times \exp\left[\left(\sum_{i=1}^{j} r_{t_i} \times \Delta t\right) - \frac{\phi^2 \cdot \sigma_r^2 + \sigma_V^2}{2} \cdot t + \sqrt{t} \times (\phi \cdot \sigma_r \cdot N_r(0,1) + \sigma_V \cdot N_V(0,1))\right]. \tag{4.3}$$

Note that the two $N_r(0,1)$'s in (4.1) and (4.3) have the same generated values from the standard normal distribution.

4.1.3 Generate values for the liquidity process

The dynamics of the liquidity process with parameter σ_γ is given by (3.5). This stochastic process is also known as the standard Brownian motion without a drift term. Given an arbitrary value of γ_0, the liquidity component at time t can be obtained by solving the stochastic differential equation. Specifically,

$$\gamma_t = \gamma_0 + \sigma_\gamma \cdot \sqrt{t} \cdot N_\gamma(0,1), \tag{4.4}$$

where $t = t_j = j \cdot \Delta t$, $\Delta t = t_j - t_{j-1} = 1/365$, $j = 1, 2, \ldots, T/\Delta t$, and $N_\gamma(0,1)$ is the generated standard normal random number.

4.1.4 Approximate the price of the CAT bond

Since the stochastic processes of r_t and γ_t are simulated on a daily basis, and they serve as the discount factor in both (3.10) and (3.12), we have

$$\begin{aligned} P_f(0,T) \approx & E\left[L \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_c > T\right] \times Pr(\tau_c > T) \\ & + E\left[a \cdot L \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_c \leq T\right] \times Pr(\tau_c \leq T), \end{aligned} \tag{4.5}$$

an approximated formula to price the CAT bond without the default risk involved.

Similarly, the expression for the approximated pricing formula when the default risk is taken into consideration becomes

$$
\begin{aligned}
&P_d(0,T)\\
&\approx E\left[L \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_d > T, \tau_c > T, V_T - C_T \geq L\right]\\
&\quad \times Pr(\tau_d > T, \tau_c > T, V_T - C_T \geq L)\\
&\quad + E\left[a \cdot L \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_d > T, \tau_c \leq T, V_T - C_T \geq a \cdot L\right]\\
&\quad \times Pr(\tau_d > T, \tau_c \leq T, V_T - C_T \geq a \cdot L)\\
&\quad + E\left[\max\{\beta \cdot (V_T - C_T),\ 0\} \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_d = T, \tau_c > T, V_T - C_T < L\right]\\
&\quad \times Pr(\tau_d = T, \tau_c > T, V_T - C_T < L)\\
&\quad + E\left[\min\{a \cdot L,\ \max[\beta \cdot (V_T - C_T),\ 0]\} \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_d = T, \tau_c \leq T, V_T - C_T < a \cdot L\right]\\
&\quad \times Pr(\tau_d = T, \tau_c \leq T, V_T - C_T < a \cdot L)\\
&\quad + E\left[\beta \cdot K^D \cdot e^{-\sum_{j=1}^{\tau_d/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t} \mid \tau_d < T < \tau_c \text{ or } \tau_d < \tau_c \leq T\right]\\
&\quad \times Pr(\tau_d < T < \tau_c \text{ or } \tau_d < \tau_c \leq T)\\
&\quad + E\left[\min\{a \cdot L \cdot e^{-\sum_{j=1}^{T/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t},\ \beta \cdot K^D \cdot e^{-\sum_{j=1}^{\tau_d/\Delta t}(r_{t_j}+\gamma_{t_j})\times\Delta t}\} \mid \tau_c \leq \tau_d < T\right]\\
&\quad \times Pr(\tau_c \leq \tau_d < T).
\end{aligned}
\tag{4.6}
$$

Note that $\tau_d/\Delta t$ is a positive integer; see Section 4.1.5 for more details.

4.1.5 Simulation procedure

The following states the steps taken in each round of simulation.

1. Generate $W_{\lambda,j}$ to get the annual occurrence intensity of catastrophic events λ_j^C (λ_j^C is constant in the entire j^{th} year) using (3.4), where j is measured in years.

2. Once the rate of occurrences per year has been determined,

   (a) calculate the trigger level K_j^C set in the CAT provisions; and

   (b) generate a vector of inter-arrival times $\{S_l : l \geq 1\}$ from the exponential distribution with mean $1/\lambda_j^C$.

3. Calculate the arrival time of the i^{th} catastrophe, $T_i = \sum_{l=1}^{i} S_l$, where i is a positive integer. $T_i/\Delta t$ is then rounded to the nearest integer I_i and let

$$\hat{T}_i = I_i \times \Delta t.$$

4. Collect the rounded arrival time $\hat{T}_i$ of the i^{th} catastrophes $i = 1, 2, \ldots,$ occurring during the CAT bond term until $\hat{T}_k$ is close to but not exceed T for some k. In other words, we find some k such that $\hat{T}_k \leq T$ and $\hat{T}_{k+1} > T$.

5. Generate k catastrophe losses $\{X_i : 1 \leq i \leq k\}$ from the lognormal distribution with mean μ_C and standard deviation σ_C.

6. Calculate the issuer's cumulative losses at each time of the triggering event, $C_{\hat{T}_i} = \sum_{j=1}^{i} X_{\hat{T}_j}$, where $i = 1, 2, \ldots, k$.

7. Determine the trigger time τ_c by finding the smallest integer $i^* \leq k$ such that $\hat{T}_{i^*} \leq T$ and $C_{\hat{T}_{i^*}} > K_j^C$ for some j. In this case, $\tau_c = \hat{T}_{i^*}$, implying that the CAT bond is triggered by or at time T. Otherwise, $\tau_c > T$.

8. Simulate the sample paths of daily interest rate under the Vasicek process by (4.1), the issuer's total assets by (4.2), and the values of liquidity process by (4.4).

9. If there exists $i' \leq k$ that meets the condition $\hat{T}_{i'} \leq T$ and $V_{\hat{T}_{i'}} - C_{\hat{T}_{i'}} < K^D$, then the default time $\tau_d = \hat{T}_{i'} = I_{i'} \times \Delta t$. Otherwise $\tau_d > T$, meaning that no default event occurs prior to or at the maturity of the CAT bond.

10. Create seven indicator variables set to 0 for seven different scenarios. Classify the result based on the conditions specified in Table 3.1 to the corresponding scenario, add 1 to that indicator variable, and calculate its present value of the payoff.

After $100,000$ times of simulation, we count the number of occurrences and add up all simulated present values of payoffs for each scenario, as per we discussed in Table 3.1. Finally, we obtain

(a) the occurring probability of each scenario, which is estimated by dividing the number of corresponding counts by $100,000$. They are the probability terms $Pr(\star)$ in (4.5) and (4.6);

(b) the discounted payoff of each scenario at time 0, which is estimated by the total corresponding discounted payoffs over $100,000$. They are the conditional expected values $E(\cdot \mid \star)$ in (4.5) and (4.6); and

(c) the price at the inception of the contract, $P_f(0,T)$ or $P_d(0,T)$, which is estimated by the average of $100,000$ simulated present values of the total payoffs. The price can also be obtained by summing the products of the probability from (a) and the conditional expected value from (b) using the same methodology as shown in (4.5) and (4.6).

Since all scenarios are mutually exclusive and exhaustive, any two of them cannot be true at the same time, and the sum of the probabilities of all disjoint scenarios is equal to 1 (all these possibilities exhaust all of the $100,000$ outcomes).

4.2 Base case

This section first gives the assumptions for the parameter values in the CAT bond provisions and the stochastic model parameters. After that, we present a numerical analysis for both default-free and defaultable CAT bond prices under the assumptions for the base case.

4.2.1 Assumptions

For the base case, all parameter values and definitions are summarized in Table 4.1. For simplicity, we assume that the bankruptcy level, K^D, and the face value of the issuer's total debts, L, are constant over time (both equal to \$100). The maturity period, T, is set to be 1 year. Further, a is set at 0.5, meaning that when the CAT bond's trigger has been pulled but the issuer is solvent, the bond holders will receive $a \cdot L$. The recovery rate β that determines the payoff at the time of default, either prior to or at the maturity date T, is set to be 0.6.

Since we do not have real data handy to estimate the parameters for the underlying stochastic models, they are chosen and adjusted based on the existing literature. For example, we refer to Lee and Yu (2002) for the parameters of the asset, interest rate dynamics, and the log-normally distributed catastrophe losses for the issuer, in which a one-year zero-coupon bond with default risk is priced. We also follow the assumption in Lee and Yu (2002) regarding the initial asset-to-liability ratio. Moreover, we refer to Lin et al. (2009) for the parameters of the doubly stochastic Poisson process that models the frequency of catastrophic events. For the liquidity process parameters, we assume the similar parameter values to those of the interest rate model since they both serve as the discount factor as shown in (3.10) and (3.12).

4.2.2 Numerical illustrations

Once the assumptions of the model/process parameter values are set for the base case, we can then compute the prices of both CAT bonds without and with default risk by following the procedure as described in Section 4.1.5.

1. CAT bonds without default risk

Table 4.1: Base case parameters

	Parameter	Value
Interest rate model		
r_0	Initial rate of return	0.05
κ	Speed of reversion	0.2
θ	Long-term mean level	0.05
σ_r	Instantaneous volatility	0.1
Asset model		
V_0	Initial asset value of the issuer	V_0/L ratio $= 1.1$
ϕ	Interest rate elasticity of asset	-3
σ_V	Volatility of credit risk	0.05
Log-normal loss model		
μ_C	Mean of the issuer's catastrophe losses	3
σ_C	Standard deviation of the issuer's catastrophe losses	0.5
CAT intensity model		
λ_0^C	Initial arrival rate of catastrophic events	2.5
μ_λ	Instantaneous drift of intensities	0.05
σ_λ	Volatility of intensities	0.01
Liquidity model		
γ_0	Initial instantaneous liquidity	0.03
σ_γ	Instantaneous volatility	0.01
Other parameters		
a	The ratio of the face value needed to be paid if the CAT bond is triggered	0.5
K^D	The bankruptcy level	100
L	The face value of the issuer's total debt	100
β	Recovery rate	0.6
T	Expiry time	1 year
Δt	Width of time step	1/365

Based on the payoff structure in (3.6), we apply the pricing formula in (4.5) to calculate the approximated price of a default-free CAT bond under each scenario. In the case that there is no default risk being considered, we only have to study two scenarios, the CAT bond is triggered before the maturity date and the triggering event does not activate the issuer's payout.

The numerical values of the three elements (conditional mean, probability and their product) that have been mentioned at the end of the simulation procedure are summarized in Table 4.2. In addition, the percentages of the price under each of the scenarios relative to the total price at the inception of the CAT bond contract are also provided.

Table 4.2: Summary under the base case without default risk

Scenario		Conditional Mean	Probability	Price	%
i.	1	93.00816	0.51485	47.88525	68.2411%
	2	45.93514	0.48515	22.28543	31.7589%
			Total	70.17068	100.0000%

The overall price is around $70.17 as shown in Table 4.2. The first scenario that the CAT bond is not triggered during the CAT bond term contributes about 68% to the total price; while the other one that represents the opposite situation occupies about 32%. The issuer does not default in both scenarios.

2. CAT bonds with default risk

We now incorporate the risk that the issuer may be insolvent and default into the pricing of a CAT bond. Table 4.3 gives the same three elements as those in Table 4.2. These results are based on the payoff structure in (3.7) − (3.9), and the price is estimated by (4.6).

Table 4.3: Summary under the base case with default risk

Scenario		Conditional Mean	Probability	Price	%
i.	1	92.30593	0.19971	18.43442	29.1464%
	2	44.34452	0.00020	0.00887	0.0140%
ii.	3	46.00891	0.12320	5.66830	8.9621%
	4	14.03417	0.00035	0.00491	0.0078%
iii.	5-1	57.76890	0.19301	11.14998	17.6291%
	5-2	58.91005	0.44508	26.21968	41.4556%
	6	45.81113	0.03845	1.76144	2.7850%
			Total	63.24759	100.0000%

As we observe from Table 4.3, the numbers of occurrence in the $100,000$ simulation runs are low for both Scenarios 2 and 4, resulting in the low ratios of the scenario price to the total price. The total price with the default risk is about \$63.25, which is cheaper than the one (\$70.17) without default risk as one may expect. The intuition for this consequence is that higher uncertainty leads to a lower price of the CAT bond, in such a way that the investors have more incentive to purchase the CAT bond.

Figure 4.1: Comparisons between the default-free and default-risky CAT bonds

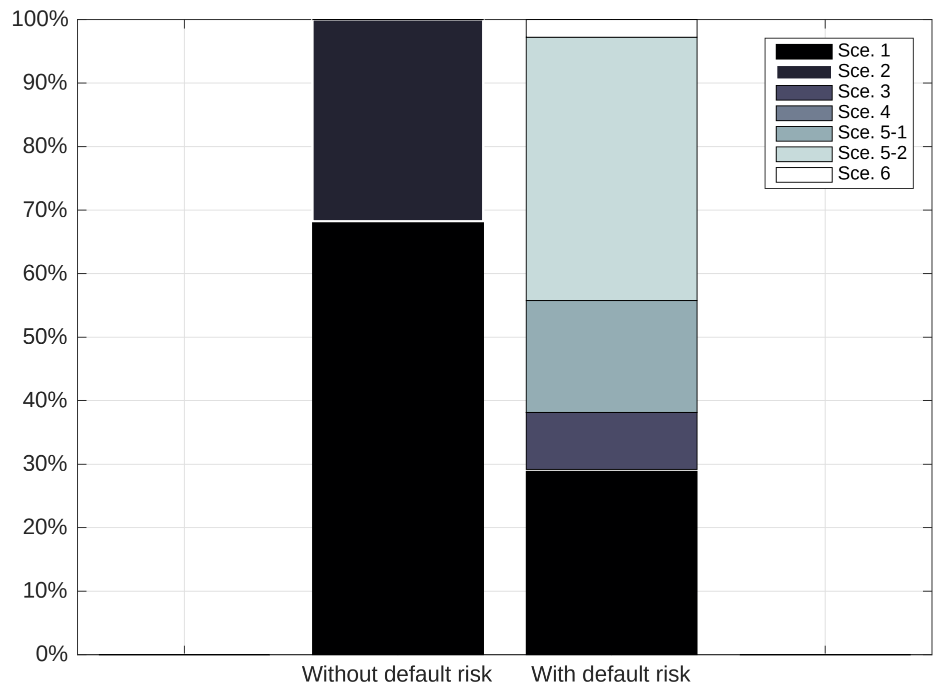


Figure 4.1 compares the allocated percentages for different scenarios under the default-free and defaultable CAT bonds. Because the issuer's default risk is taken into account, the percentages for Scenarios 1 and 2 for the case of a CAT bond without default risk are then distributed into seven scenarios for the case of a CAT bond with default risk. More detailed, with the consideration of default risk, the price for Scenario 1 to the total price goes down remarkably from 68.2411% to 29.1464%. The notable influence on the percentage for Scenario 2 (down to 0.0140% from 31.7589%) can also be observed in Figure 4.1.

4.3 Sensitivity analysis

This section analyzes the sensitivity of the price at the issuing date of a defaultable CAT bond, $P_d(0,T)$, to the key parameters of each of the stochastic processes, as well as the remaining parameters in the CAT bond provisions.

In the following subsections, each figure displays three groups of eight bars, where each group represents the prices for a value of some key parameter. Also, the first seven bars in a group represent the prices for the seven different scenarios with the same order as Table 3.1, while the last bar gives the total price of the defaultable CAT bond. For a tiny bar, we add a numerical value to specify the corresponding price.

4.3.1 Effect of changes in the interest rate model parameters

The purpose of this sensitivity test is to study the effect of interest rate risk on the CAT bond price. Since we assume the CAT bond expires in one year in this project, the changes in some parameter values will not affect the price significantly. As a result, we do not include the impacts of the initial rate of return, r_0, and the long term mean level, θ, on the bond price. Lastly, a brief summary of the effects are provided.

1. Volatility of interest rate risk, σ_r

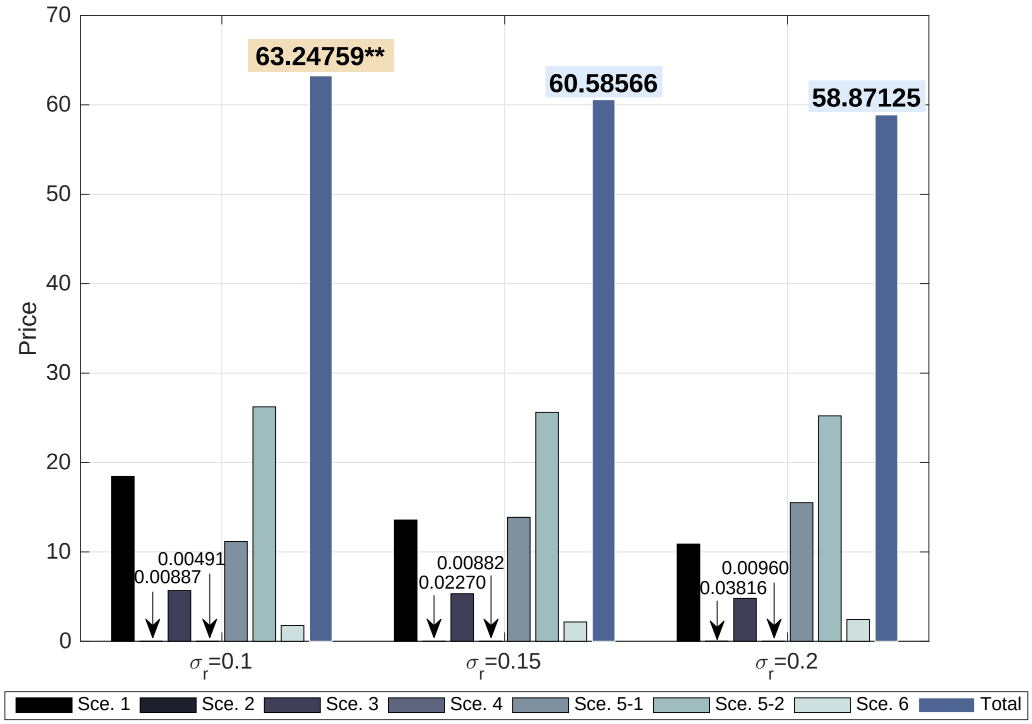
   Figure 4.2 plots the result at three different values of σ_r. It shows that at higher parameter value, as the amount of randomness of the process increases, the overall bond price drops. For example, when σ_r changes from 0.1 to 0.2, the total price of the CAT bond falls from around \$63.25 to \$58.87.

Table 4.4: Probabilities under each scenario for three different values of σ_r

Scenario		Probability			
		$\sigma_r = 0.1^{**}$	$\sigma_r = 0.15$	$\sigma_r = 0.2$	Trend
i.	1	0.19971	0.14717	0.11812	↓
	2	0.00020	0.00051	0.00085	↑
ii.	3	0.12320	0.12973	0.13171	↑
	4	0.00035	0.00092	0.00145	↑
iii.	5-1	0.19301	0.23902	0.26609	↑
	5-2	0.44508	0.43530	0.42844	↓
	6	0.03845	0.04735	0.05334	↑

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend.

Figure 4.2: CAT bond prices for three different values of σ_r



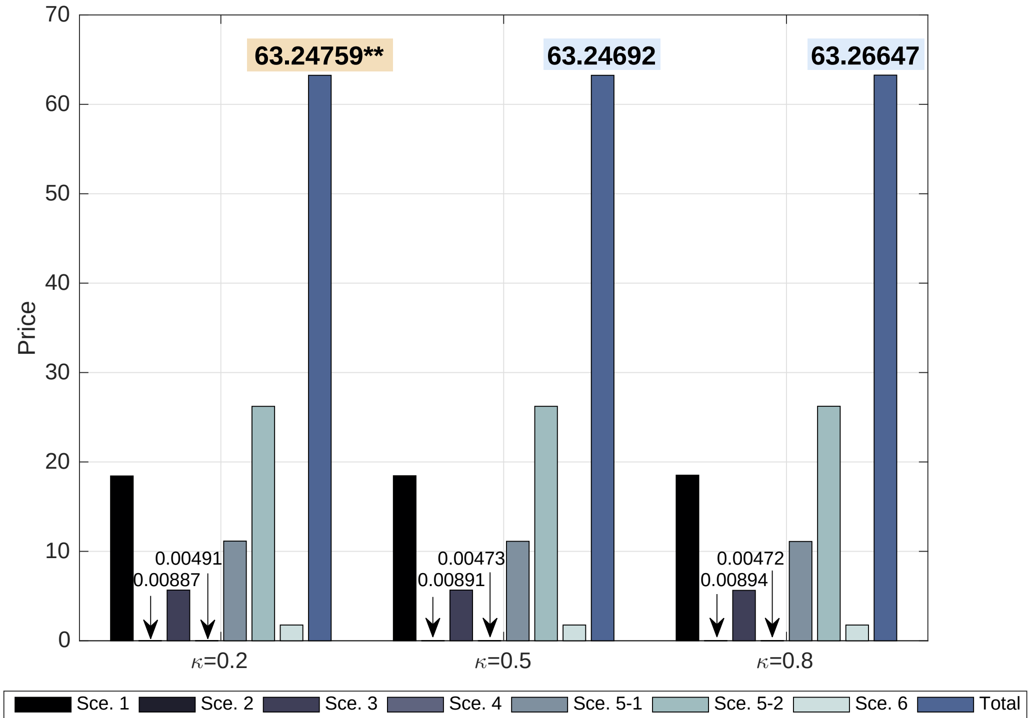
$**$: base case.

Table 4.4 gives the detailed changes in the probability under each of three cases and each of seven scenarios. Recall that in Chapter 3 we incorporate the interest rate risk into the asset dynamics, so the growing volatility affects not only the outcomes of the simulated interest rates but also the CAT bond issuer's total assets. Because of that, the more volatile the value of issuer's assets is, the higher the uncertainty is for the bond holders to get their money back when the triggering event occurs and incurs a large amount of losses to the issuer. Based on these reasons, it is not surprising that the probability under Scenario 1 displays a decreasing trend when the value of σ_r gets higher, as presented in Table 4.4. Although most of the scenarios present an increasing trend in probability, however, we can see from Figure 4.2 that the decrease in price under Scenario 1 is much more than the increase in price under other scenarios. Consequently, the overall bond price is lower as the interest rate volatility σ_r rises.

2. Speed of reversion, κ

Figure 4.3 exhibits the prices under each scenario and the total price for three different values of κ. This figure shows that the increase in κ causes a tiny growth in the bond price. In addition, Table 4.5 displays the details of changes in the probability under each of the scenarios.

Figure 4.3: CAT bond prices for three different values of κ



$**$: base case.

Table 4.5: Probabilities under each scenario for three different values of κ

Scenario		Probability			
		$\kappa = 0.2^{**}$	$\kappa = 0.5$	$\kappa = 0.8$	Trend
i.	1	0.19971	0.19988	0.20051	$\uparrow$
	2	0.00020	0.00020	0.00020	$-$
ii.	3	0.12320	0.12333	0.12288	$\uparrow$
	4	0.00035	0.00034	0.00034	$\downarrow$
iii.	5-1	0.19301	0.19271	0.19253	$\downarrow$
	5-2	0.44508	0.44513	0.44520	$\uparrow$
	6	0.03845	0.03841	0.03834	$\downarrow$

$**$: base case; $\uparrow$: increasing trend; $\downarrow$: decreasing trend; $-$: no significant trend.

The main reason why we study the way in which the CAT bond value varies when κ increases is to compare the influences of σ_r and κ on the bond prices. For the Vasicek model, the variance in the long run is $\sigma_r/(2\kappa)$. In other words, the future interest rate will tend to move towards the long term equilibrium level, θ, with such a variance. Intuitively, increasing κ implies the larger mean-reverting force that drifts the process towards its long term mean faster over time. In contrast, a rising σ_r increases the randomness. From another perspective, it is clearly to see that the value of $\sigma_r/(2\kappa)$ increases with σ_r but decreases with κ. Thus, we can expect that the CAT bond becomes more expensive with a lower degree of randomness. The prices at time 0 under three different levels of σ_r and κ are displayed in Table 4.6.

Table 4.6: Overall effects of κ and σ_r on the CAT bond price, $P_d(0,T)$

κ	$P_d(0,T)$	σ_r	$P_d(0,T)$
0.2**	63.24759	0.10**	63.24759
0.5	63.24692	0.15	60.58566
0.8	63.26647	0.20	58.87125
Overall effect	Increase		Decrease

** : base case.

Table 4.6 shows an opposite effect of these two parameters on the CAT bond price, which supports the argument we just mentioned. For example, $P_d(0,T)$ grows slightly from about \$63.25 to \$63.27 when κ increases from 0.2 to 0.8, whereas $P_d(0,T)$ drops by around \$4.38 when σ_r increases from 0.1 to 0.2.

To sum up, the general pattern for changes in the Vasicek interest rate model parameters are summarized in Table 4.7.

Table 4.7: Impacts of the parameters of the Vasicek model on $P_d(0,T)$

Increase in parameter value	Effect on $P_d(0,T)$
σ_r	$\downarrow$
κ	$\uparrow$

$\uparrow$: increasing trend; $\downarrow$: decreasing trend.

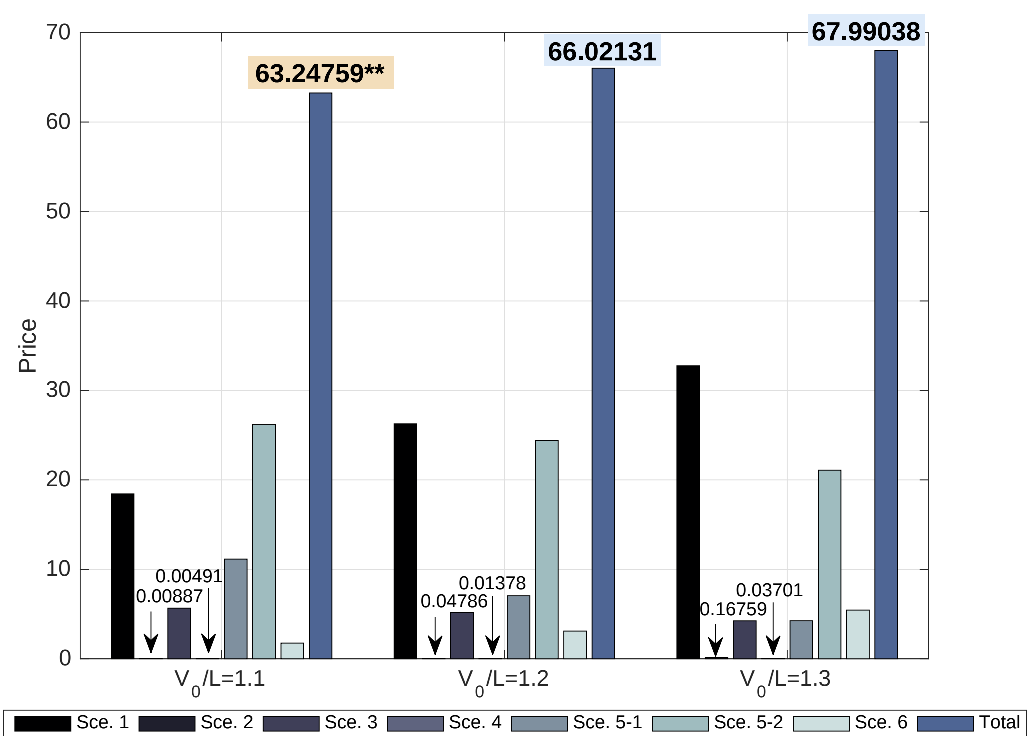
4.3.2 Effect of changes in the asset model parameters

First we test the sensitivity of the defaultable CAT bond price with respect to each of the parameters in the asset model. At the end of this subsection, a brief summary of effects is provided.

1. Asset-to-liability ratio, V_0/L

   The defaultable CAT bond prices for three different values of asset-to-liability ratio are given in Figure 4.4. The figure shows that as V_0/L increases, the CAT bond becomes more expensive due to the issuer's growing capability of repaying money to the CAT bond holders. For instance, the price of the bond rises from about \$63.25 to \$67.99 when we increase V_0/L from 1.1 to 1.3.

   Figure 4.4: CAT bond prices for three different asset-to-liability ratios

   ** : base case.

   Furthermore, if we look into each of the scenarios, we can see that Scenarios 3, 5-1 and 5-2 show a downward trend since it is less likely that the issuer defaults even when a catastrophe occurs during the risk period of the CAT bond. Or equivalently, the investors expect to get most or all of their face value back when the CAT bond matures. Either of these reasons makes the bond more valuable. Conversely, for the

Table 4.8: Probabilities under each scenario for three different values of V_0/L

Scenario		Probability			
		$V_0/L = 1.1^{**}$	$V_0/L = 1.2$	$V_0/L = 1.3$	Trend
i.	1	0.19971	0.28404	0.35365	↑
	2	0.00020	0.00106	0.00368	↑
ii.	3	0.12320	0.10950	0.08829	↓
	4	0.00035	0.00098	0.00244	↑
iii.	5-1	0.19301	0.12238	0.07398	↓
	5-2	0.44508	0.41438	0.35926	↓
	6	0.03845	0.06766	0.11870	↑

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend.

rest of the scenarios, the prices increase with the issuer's asset-to-liability ratio. The same pattern can also be found in Table 4.8 which provides the overall changes in the probability under each scenario.

2. Volatility of credit risk, σ_V

Figure 4.5 illustrates the effect of σ_V on the CAT bond price. This figure shows a decreasing trend in the bond prices when the volatility of credit risk rises. It is because a larger volatility implies a higher degree of variation in the asset value over time, and hence leads to a greater likelihood of the issuer going bankrupt no matter whether the specified trigger conditions are met or not, which causes the bond's value to go down. For example, the bond price drops by about \$2 when the value of σ_V increases from 0.05 to 0.3.

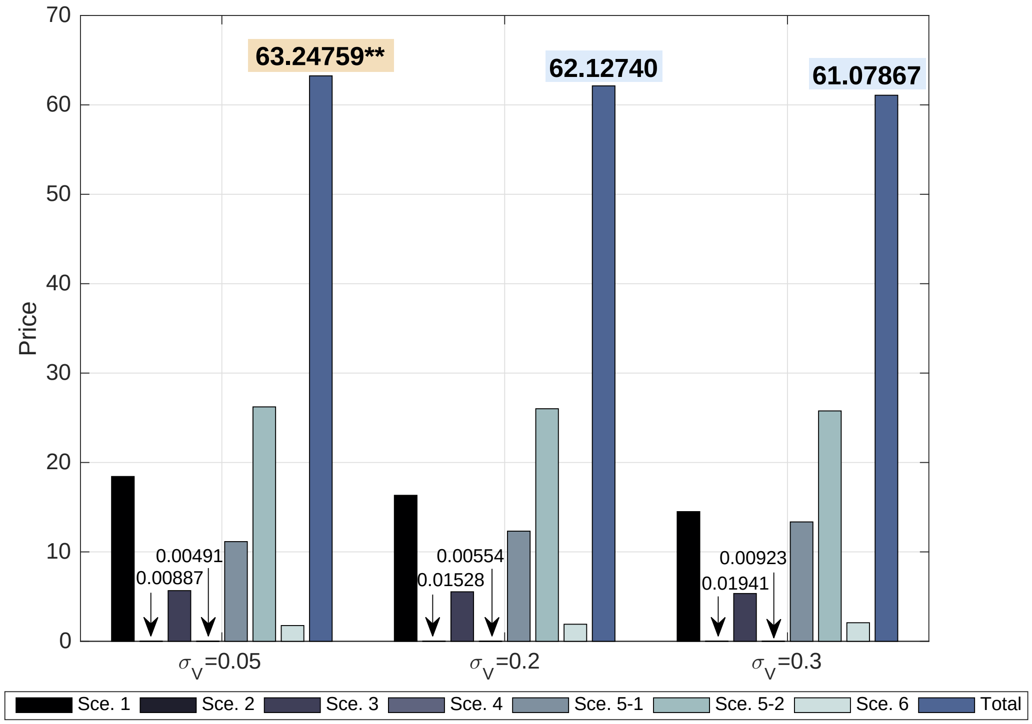
In particular, we can observe that under Scenario 1 where neither an event of default occurs nor the CAT bond is triggered during the maturity period of the bond, the CAT bond price decreases significantly as σ_V increases. The prices under Scenario 5-2 also show a relatively inconsiderable downward trend for increasing σ_V values. The reverse effect can be found in the remaining scenarios.

Table 4.9 provides the summary of changes in probability. According to Table 4.9, the patterns for the changes in the probability under different levels of volatility, σ_V, are identical to those in the CAT bond price.

3. Interest rate elasticity of asset, ϕ

Figure 4.6 gives the CAT bond prices with respect to three different ϕ values. When

Figure 4.5: CAT bond prices for three different values of σ_V



∗∗ : base case.

Table 4.9: Probabilities under each scenario for three different values of σ_V

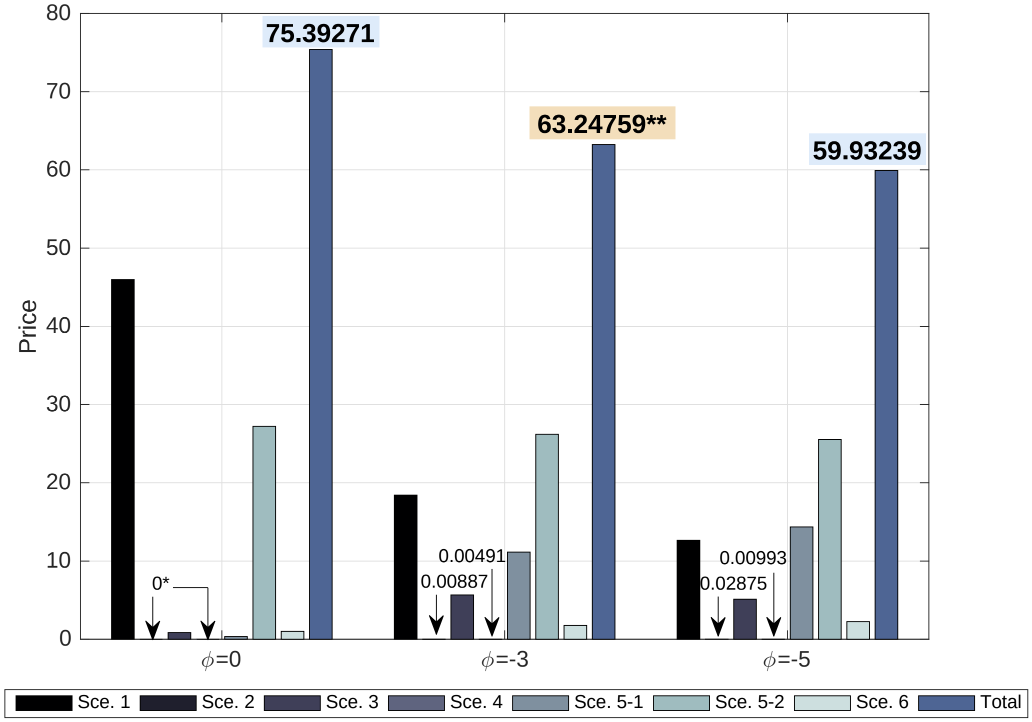
Scenario		Probability			
		$\sigma_V = 0.05^{**}$	$\sigma_V = 0.2$	$\sigma_V = 0.3$	Trend
i.	1	0.19971	0.17681	0.15699	↓
	2	0.00020	0.00034	0.00043	↑
ii.	3	0.12320	0.12584	0.12782	↑
	4	0.00035	0.00053	0.00084	↑
iii.	5-1	0.19301	0.21327	0.23111	↑
	5-2	0.44508	0.44158	0.43754	↓
	6	0.03845	0.04163	0.04527	↑

∗∗ : base case; ↑ : increasing trend; ↓ : decreasing trend.

ϕ falls from 0 to -5, the price decreases drastically by over \$15. As explained in Lee and Yu (2002), a higher absolute value of interest rate elasticity creates a higher volatility of issuer's default risk. Our result is consistent with the pattern as in Lee and Yu (2002). Table 4.10 provides more detailed impacts of three different values of

ϕ on the occurrence of each of the scenarios with $100,000$ simulation runs.

Figure 4.6: CAT bond prices for three different values of ϕ



$*$: this scenario does not occur in the $100,000$ runs; $**$: base case.

Table 4.10: Probabilities under each scenario for three different values of ϕ

Scenario		Probability			
		$\phi = 0$	$\phi = -3^{**}$	$\phi = -5$	Trend
i.	1	0.49582	0.19971	0.13669	↓
	2	0.00000	0.00020	0.00063	↑
ii.	3	0.01439	0.12320	0.13076	–
	4	0.00000	0.00035	0.00109	↑
iii.	5-1	0.00571	0.19301	0.24847	↑
	5-2	0.46200	0.44508	0.43334	↓
	6	0.02208	0.03845	0.04902	↑

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend; – : no significant trend.

To conclude, the results for the overall changes in asset model parameters are summarized in Table 4.11.

Table 4.11: Impacts of the parameters of the asset model on $P_d(0,T)$

Increase in parameter value	Effect on $P_d(0,T)$
V_0/L	$\uparrow$
σ_V	$\downarrow$
ϕ	$\downarrow$

$\uparrow$: increasing trend; $\downarrow$: decreasing trend.

4.3.3 Effect of changes in the aggregate loss model parameters

Here we first examine the sensitivity of $P_d(0,T)$ under sixteen different combinations of parameters in the aggregate loss model. We focus on the parameters μ_C, λ_C, μ_λ and σ_λ other than the initial value (λ_0^C) of the aggregate loss process because of the same reason that the CAT bond matures in one year, and the effect of a change in λ_0^C is not significant as one may expect. The following Table 4.12 gives an overview of the CAT bond price under the compound doubly stochastic Poisson process with $\lambda_0^C = 2.5$.

Table 4.12: CAT bond prices under the aggregate loss model

		$(\mu_\lambda, \sigma_\lambda)$			
σ_C	μ_C	$(0.05, 0.01)$	$(0.05, 0.1)$	$(0.5, 0.01)$	$(0.5, 0.1)$
1	2	61.92155	62.52160	61.31362	61.68471
	3	62.59355	63.14338	61.64132	62.00858
2	2	63.16100	63.59509	61.46369	61.79564
	3	63.33969	63.76677	61.61973	61.94469

The price under the base case is \$63.24759 with parameter values $(\lambda_0^C, \mu_\lambda, \sigma_\lambda, \mu_C, \sigma_C) = (2.5, 0.05, 0.01, 3, 0.5)$.

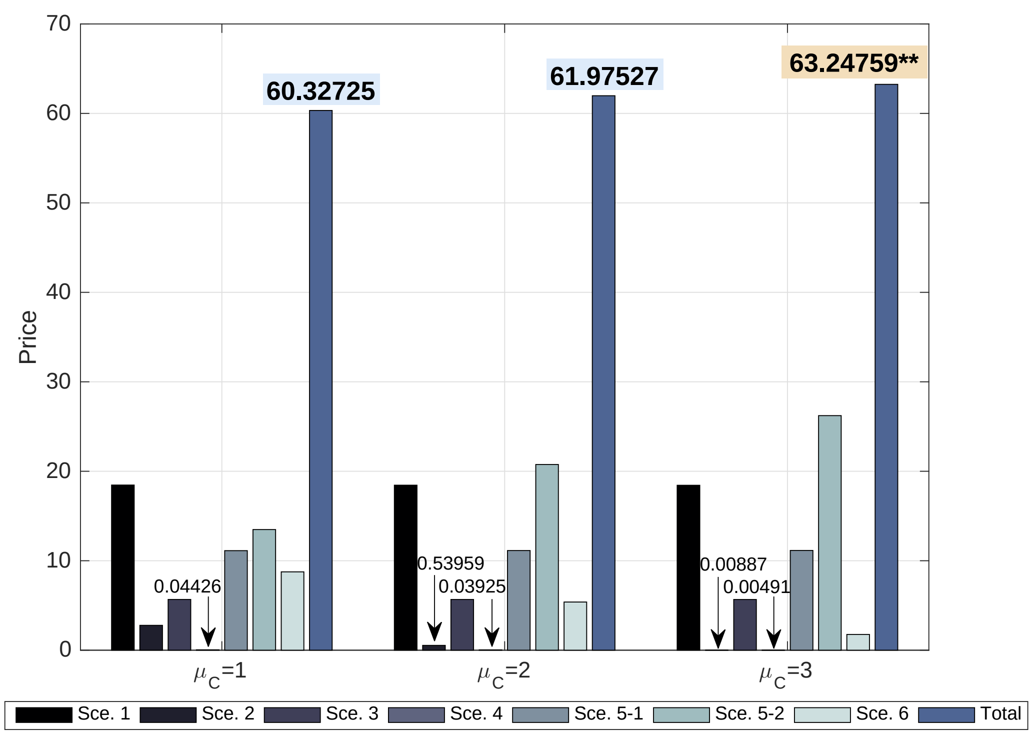
From Table 4.12, we note that the overall bond price becomes cheaper when the instantaneous growth rate (μ_λ) of catastrophic frequency increases, while an increase in the volatility (σ_λ) appears to affect the CAT bond price in an opposite way. This table also reveals that increasing values of both μ_C and σ_C result in a higher value of the CAT bond.

Similarly as before, we discuss the details under each scenario for different values of parameters μ_C, σ_C, μ_λ and σ_λ below.

1. Log-normal loss model - μ_C

   Figure 4.7 diagrams the impact of μ_C on the CAT bond price. The figure shows an upward trend in the price of the CAT bond as the value of μ_C increases, in spite of that the overall simulated losses are larger at the same time. For example, the price grows by roughly \$3 when μ_C rises from 1 to 3. One of the reasons is that the annual trigger level, K_j^C, increases with the mean of catastrophic loss. A higher trigger threshold indicates that the CAT bond is less likely to be triggered when a catastrophic event occurs, and thus makes the CAT bond worth investing from the viewpoint of the investors. Table 4.13 displays the patterns under each of the seven scenarios.

   Figure 4.7: CAT bond prices for three different values of μ_C

   ∗∗ : base case.

   Particularly, the numbers of occurrences in the 100,000 simulation runs increase under both Scenarios 5-1 and 5-2 as μ_C increases, meaning that the event of premature default is more likely to happen due to lager size of losses no matter whether the

Table 4.13: Probabilities under each scenario for three different values of μ_C

Scenario		Probability			
		$\mu_C = 1$	$\mu_C = 2$	$\mu_C = 3^{**}$	Trend
i.	1	0.20000	0.19982	0.19971	↓
	2	0.06061	0.01180	0.00020	↓
ii.	3	0.12331	0.12327	0.12320	↓
	4	0.00180	0.00180	0.00035	↓
iii.	5-1	0.19261	0.19283	0.19301	↑
	5-2	0.23088	0.35326	0.44508	↑
	6	0.19079	0.11722	0.03845	↓

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend.

trigger level is pulled or not. This supports the argument that although a higher μ_C generally produces the greater amount of losses that makes the issuer more possible to be insolvent, it meanwhile puts up the trigger barrier (K_j^C) that reduces the possibility of losses exceeding the barrier.

2. Log-normal loss model - σ_C

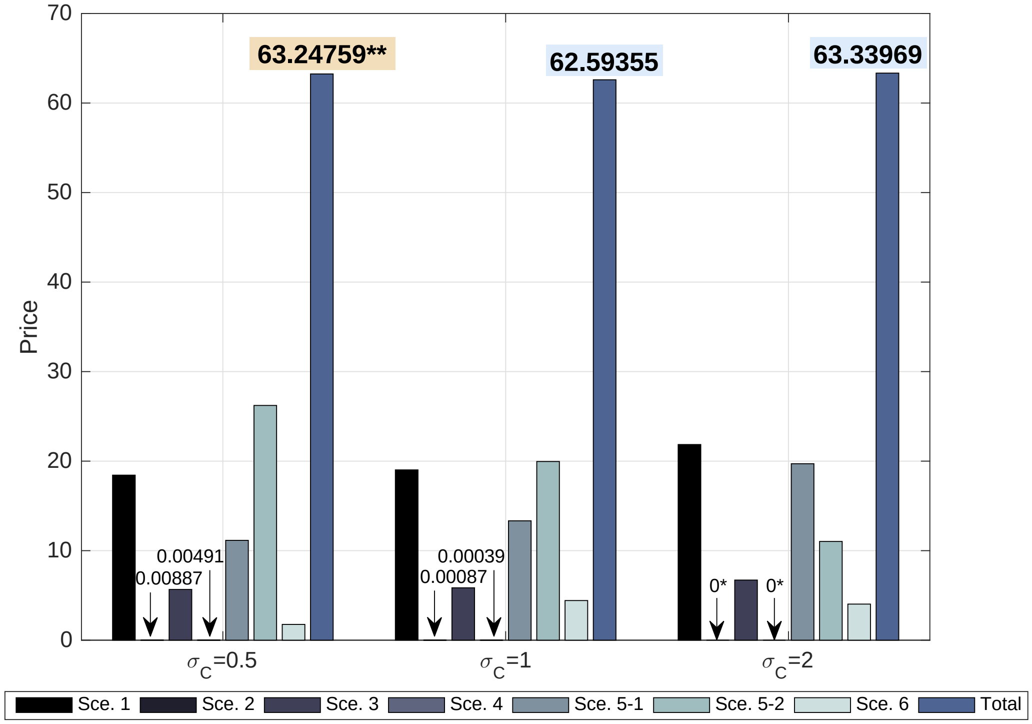
   In this example, it is interesting to see that there is actually no specific trend in the CAT bond price when we increase the value of σ_C as shown in Figure 4.8. Intuitively, the variations in losses generated from the log-normal distribution are comparatively large when the loss standard deviation is high. Therefore, it probably has more extreme values, either larger or smaller size of losses. Table 4.14 presents the detailed changes under each of the scenarios.

Table 4.14: Probabilities under each scenario for three different values of σ_C

Scenario		Probability			
		$\sigma_C = 0.5^{**}$	$\sigma_C = 1$	$\sigma_C = 2$	Trend
i.	1	0.19971	0.20613	0.23716	↑
	2	0.00020	0.00002	0.00000	↓
ii.	3	0.12320	0.12727	0.14621	↑
	4	0.00035	0.00006	0.00000	↓
iii.	5-1	0.19301	0.23082	0.34118	↑
	5-2	0.44508	0.33906	0.18747	↓
	6	0.03845	0.09664	0.08798	−

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend; − : no significant trend.

Figure 4.8: CAT bond prices for three different values of σ_C



$*$: this scenario does not occur in the $100,000$ runs; $**$: base case.

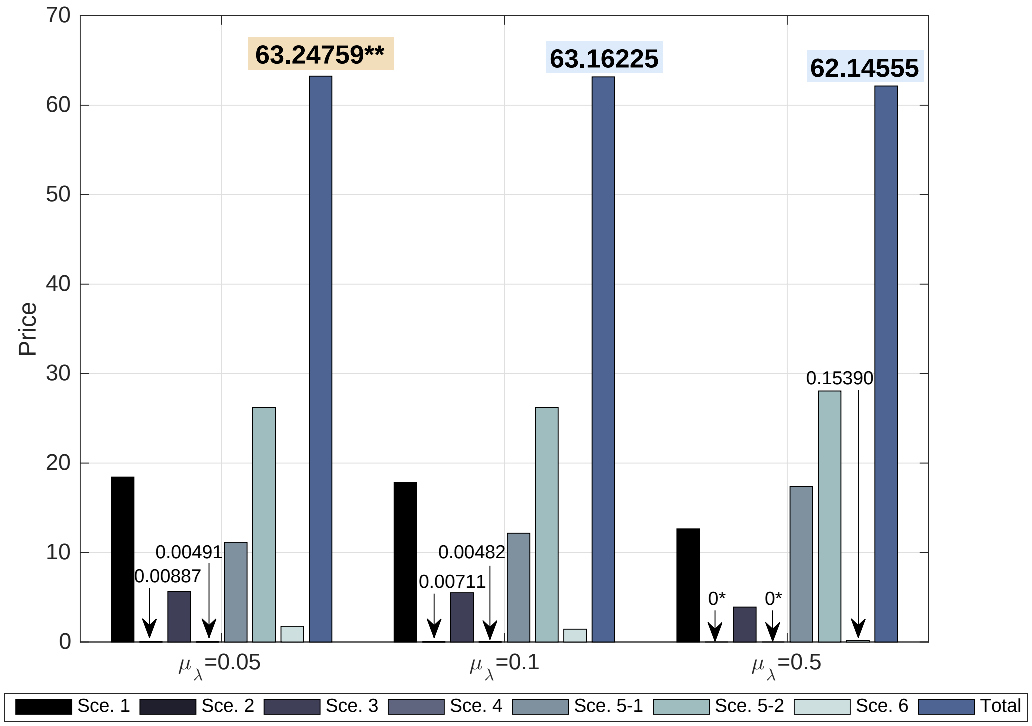
3. CAT intensity model - μ_λ

   Figure 4.9 shows how the CAT bond price varies with three different values of μ_λ. We observe from this figure that the price goes down by around \$1 when the value of μ_λ rises from 0.05 to 0.5. Recall from (3.4) that the annual arrival rate of catastrophic events, λ_j^C, increases with μ_λ. When catastrophic events arrive at a higher rate, it implies that the time between events has a lower mean, $1/\lambda_j^C$. That is, a higher λ_j^C indicates that the catastrophic events occur more frequently during the risk period of the CAT bond. Hence, either the value of an issuer's assets less the amount of aggregate losses falls below the bankruptcy level or the issuer's aggregate loss exceeds the trigger level is more likely to occur. Because of this reason, the decline in the overall bond price can be treated as a compensation for higher uncertainty. More detailed changes in the probability under three different values of μ_λ are reported in Table 4.15.

4. CAT intensity model - σ_λ

   Now we study the effects of three different choices of the volatility σ_λ on the price of the CAT bond. Figure 4.10 illustrates the results. We note that the CAT bond

Figure 4.9: CAT bond prices for three different values of μ_λ



$*$: this scenario does not occur in the 100,000 runs; $**$: base case.

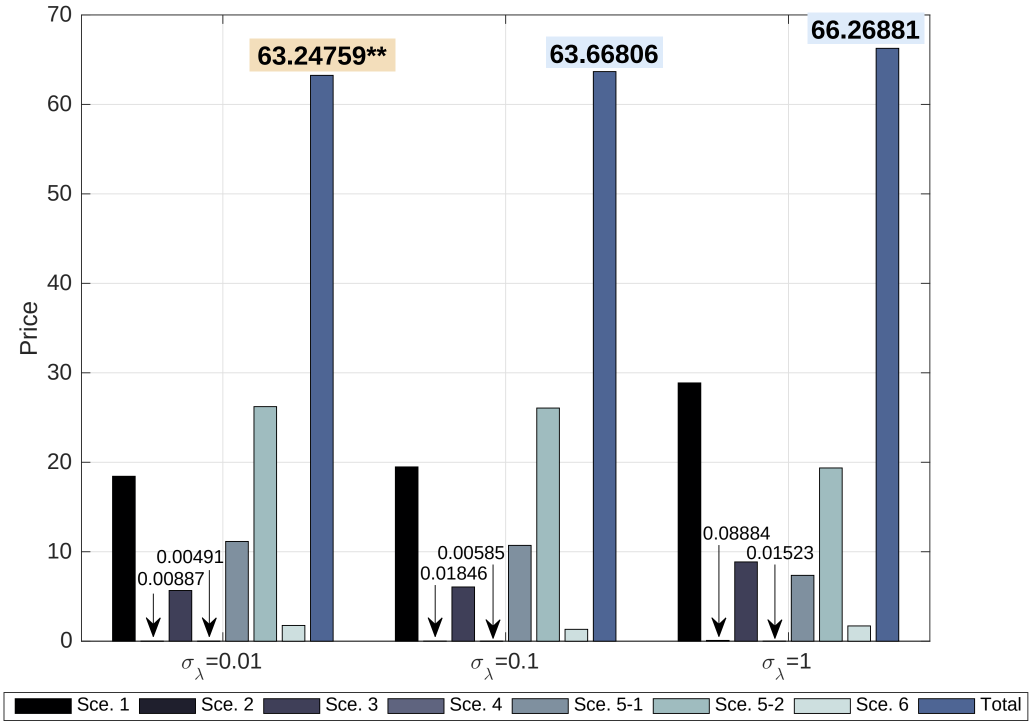
Table 4.15: Probabilities under each scenario for three different values of μ_λ

Scenario		Probability			
		$\mu_\lambda = 0.05^{**}$	$\mu_\lambda = 0.1$	$\mu_\lambda = 0.5$	Trend
i.	1	0.19971	0.19317	0.13694	↓
	2	0.00020	0.00016	0.00000	↓
ii.	3	0.12320	0.11967	0.08493	↓
	4	0.00035	0.00029	0.00000	↓
iii.	5-1	0.19301	0.21056	0.30026	↑
	5-2	0.44508	0.44480	0.47451	−
	6	0.03845	0.03135	0.00336	↓

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend; − : no significant trend.

price goes up with σ_λ based on the parameter values we select. For example, the CAT bond price increases from about \$63.25 to \$66.27 when the volatility of catastrophic intensity rises from 0.01 to 1. The details regarding the changes in the probability are provided in Table 4.16.

Figure 4.10: CAT bond prices for three different values of σ_λ



$**$: base case.

Table 4.16: Probabilities under each scenario for three different values of σ_λ

Scenario		Probability			
		$\sigma_\lambda = 0.01^{**}$	$\sigma_\lambda = 0.1$	$\sigma_\lambda = 1$	Trend
i.	1	0.19971	0.21098	0.31290	↑
	2	0.00020	0.00041	0.00194	↑
ii.	3	0.12320	0.13171	0.19288	↑
	4	0.00035	0.00048	0.00096	↑
iii.	5-1	0.19301	0.18554	0.12716	↓
	5-2	0.44508	0.44194	0.32706	↓
	6	0.03845	0.02894	0.03710	−

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend; − : no significant trend.

At last, based on our numerical illustrations, we conclude this section by summarizing the impacts of the aggregate loss model parameters μ_C, σ_C, μ_λ and σ_λ on the CAT bond price in Table 4.17.

Table 4.17: Impacts of the aggregate loss parameters on $P_d(0,T)$

	Increase in parameter value	Effect on $P_d(0,T)$
Log-normal loss severity distribution	μ_C	↑
	σ_C	−
Arrival process for catastrophic events	μ_λ	↓
	σ_λ	↑

↑ : increasing trend; ↓ : decreasing trend; − : no significant trend.

4.3.4 Effect of changes in the liquidity model parameters

We now investigate the CAT bond prices with various choices of parameter values in the liquidity dynamics. Table 4.18 provides the CAT bond prices under this stochastic process with different parameter values. Here we discuss only the general effect instead of the details. The reason is as discussed in Chapter 3; since γ_t plays the role in the discount factor in (3.12), the patterns for the changes in price and probability under each of the scenarios can be expected to be similar.

The results in Table 4.18 show that the CAT bond price $P_d(0,T)$ decreases as γ_0 increases. When the initial value (γ_0) of the liquidity process goes up, the value (γ_t) of the liquidity process increases, and further leads to a lower CAT bond price. Therefore, we can regard the decrease in price as the compensation for the CAT bond investors who bear a higher liquidity risk. On the other hand, the choice of σ_γ has relatively lower impacts on the CAT bond price. A higher σ_γ value slightly increases the CAT bond price since it causes the value (γ_t) of the liquidity process more volatile and sometimes may produce negative values, which finally results in a higher present value of the payoff. Take $\gamma_0 = 0.01$ for illustration, Table 4.19 presents the difference between the CAT bond prices for two different values of σ_γ. The table shows that the total of positive differences is greater than the absolute value of that of the negative differences, and hence the overall price rises when σ_γ increases. The same situation occurs for different choices of γ_0.

In conclusion, the results for the changes in parameter value under liquidity dynamics are summarized in Table 4.20.

Table 4.18: CAT bond prices under the liquidity process

σ_γ	γ_0	$P_d(0,T)$
0.1	0.01	64.01466
	0.05	62.50047
1	0.01	64.07041
	0.05	62.55387
2	0.01	64.17471
	0.05	62.65441

Table 4.19: Prices under each scenario with $\gamma_0 = 0.01$ and two values of σ_γ

Scenario		Price		
		$\sigma_\gamma = 1$	$\sigma_\gamma = 2$	Difference*
i.	1	18.83424	18.88754	+0.05330
	2	0.00895	0.00887	−0.00008
ii.	3	5.80487	5.83511	+0.03024
	4	0.00498	0.00495	−0.00003
iii.	5-1	11.28429	11.29909	+0.01480
	5-2	26.33087	26.33003	−0.00084
	6	1.80222	1.80912	+0.00691
			Overall	+0.10430

∗: Difference = (the price for $\sigma_\gamma = 2$) − (the price for $\sigma_\gamma = 1$).

Table 4.20: Impacts of the parameters of the liquidity model on $P_d(0,T)$

Increase in parameter value	Effect on $P_d(0,T)$
γ_0	↓
σ_γ	↑

↑ : increasing trend; ↓ : decreasing trend.

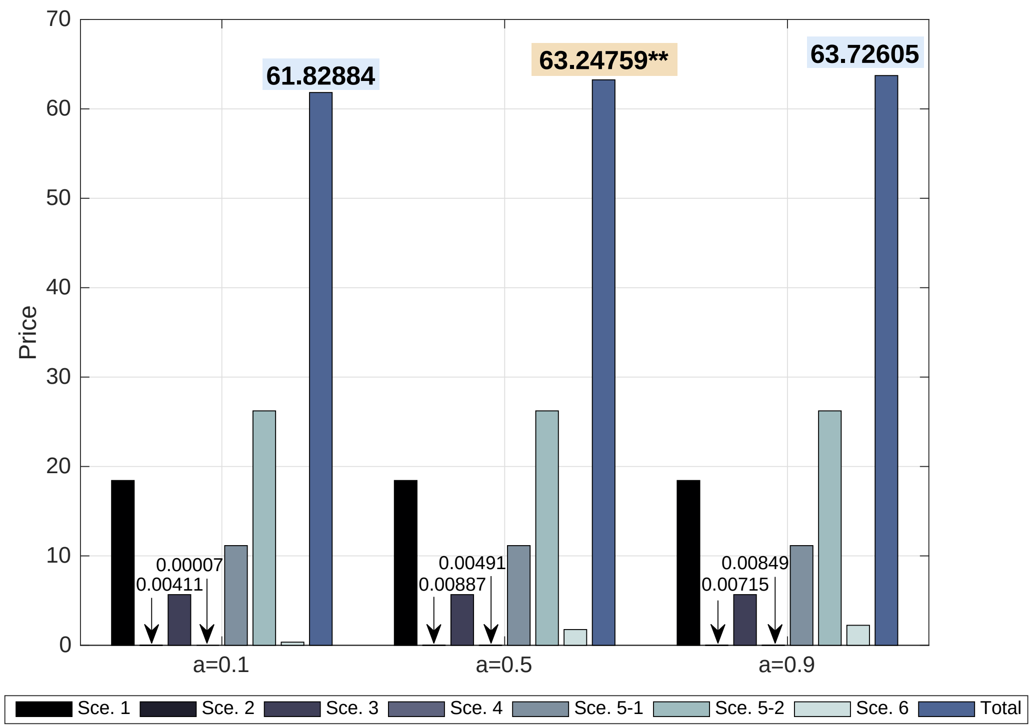
4.3.5 Effect of changes in other parameters

Finally, the last sensitivity analysis is to study the effects of the remaining parameters such as a, K^D, and β in the CAT bond provisions by examining how the CAT bond price varies with these parameters.

1. The ratio of the face value needed to be paid if the CAT bond is triggered, a

Figure 4.11 diagrams the CAT bond prices at different levels of a. If the issuer promises to repay most of the face value to the CAT bond's investors when the CAT bond is triggered before its maturity, then the CAT bond is considered more valuable. Or else the CAT bond loses its attractiveness, and the issuer has to reduce the price in order to attract the investors to purchase the CAT bond. Figure 4.11 demonstrates that the CAT bond price goes up from roughly \$61.83 to \$63.73 when a increases from 0.1 to 0.9.

Figure 4.11: CAT bond prices for three different values of a



$**$: base case.

Changing the value of a affects the payoff rather than the probability, so we summarize the changes in percentage, relative to the total price, for each scenario in the Trend column of Table 4.21. We observe from Table 4.21 that Scenarios 2, 4 and 6 show an increasing trend in the percentage since the payoffs under these three scenarios involve a. In contrast, the remaining scenarios show a reverse pattern.

2. The issuer's bankruptcy level, K^D

Overall, the relationship between the CAT bond price, $P_d(0, T)$, and the bankruptcy

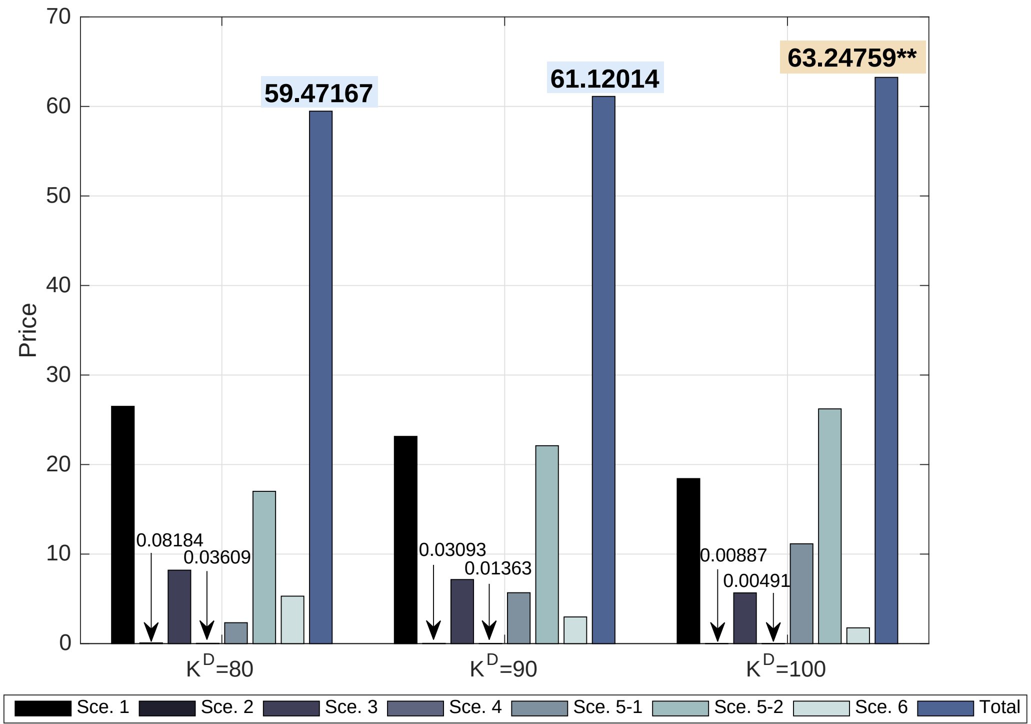
Table 4.21: Percentages of total price for three different values of a

Scenario		Percentage of total price			
		$a = 0.1$	$a = 0.5^{**}$	$a = 0.9$	Trend
i.	1	29.8152%	29.1464%	28.9276%	↓
	2	0.0066%	0.0140%	0.0112%	↑
ii.	3	9.1677%	8.9621%	8.8948%	↓
	4	0.0001%	0.0078%	0.0133%	↑
iii.	5-1	18.0336%	17.6291%	17.4967%	↓
	5-2	42.4069%	41.4556%	41.1444%	↓
	6	0.5698%	2.7850%	3.5120%	↑

$**$: base case; ↑ : increasing trend; ↓ : decreasing trend.

level, K^D, is illustrated in Figure 4.12.

Figure 4.12: CAT bond prices for three different values of K^D



$**$: base case.

When the issuer's total assets less total losses incurred by a triggering event drops below the bankruptcy level K^D, it is considered as an event of default. A higher value of K^D exposes the issuer to a higher degree of default risk since it is easier for the issuer to meet the condition of bankruptcy. As a consequence, it results in a lower overall price of the CAT bond. From the perspective of the investors, a cheaper price is regarded as an incentive for them to invest in the CAT bonds with a higher insolvent risk of the issuer. Although this intuition seems reasonable, however, it is worth noting that the price rises from around \$59.47 to \$63.25 when K^D increases from 80 to 100 as presented in Figure 4.12. To investigate this, let us look into the changes in each single scenario given in Table 4.22.

Table 4.22: Summary of the results for three different values of K^D

		Scenario						
		i.		ii.		iii.		
K^D		1	2	3	4	5-1	5-2	6
80	Conditional Mean	92.38634	45.21596	46.03907	13.36784	45.92860	46.95558	45.18127
	Probability	0.28689	0.00181	0.17820	0.00270	0.05083	0.36224	0.11733
	Price	26.50472	0.08184	8.20416	0.03609	2.33455	17.00919	5.30112
	%	44.5670%	0.1376%	13.7951%	0.0607%	3.9255%	28.6005%	8.9137%
90	Conditional Mean	92.33586	44.82995	46.04046	13.36324	51.78927	52.95908	45.87137
	Probability	0.25069	0.00069	0.15547	0.00102	0.10976	0.41743	0.06494
	Price	23.14768	0.03093	7.15791	0.01363	5.68439	22.10671	2.97889
	%	37.8724%	0.0506%	11.7112%	0.0223%	9.3004%	36.1693%	4.8738%
100**	Conditional Mean	92.30593	44.34452	46.00891	14.03417	57.76890	58.91005	45.81113
	Probability	0.19971	0.00020	0.12320	0.00035	0.19301	0.44508	0.03845
	Price	18.43442	0.00887	5.66830	0.00491	11.14998	26.21968	1.76144
	%	29.1464%	0.0140%	8.9621%	0.0078%	17.6291%	41.4556%	2.7850%
	Trend	↓	↓	↓	↓	↑	↑	↓

** : base case; ↑ : increasing trend; ↓ : decreasing trend.

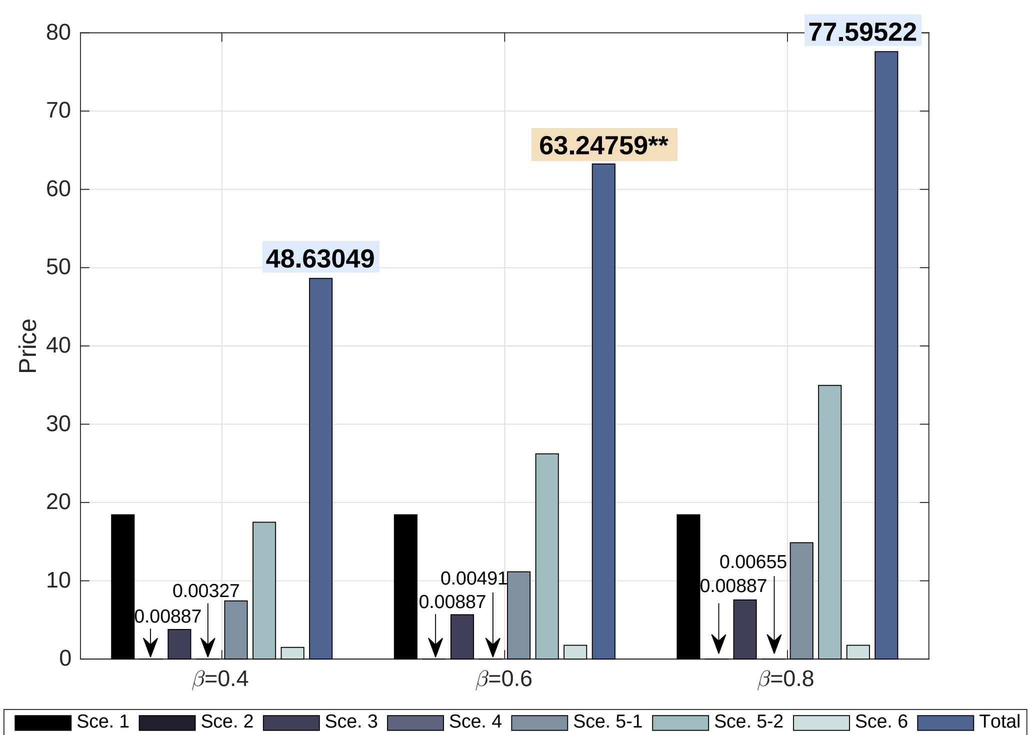
In Table 4.22, we provide the conditional mean, probability, scenario price and the percentage for each scenario under three different values of K^D. Observe from this table that Scenario 1 and Scenario 5-2 take up roughly 70% of the total price. Despite the fact that the probabilities and percentages for most scenarios decrease as K^D increases, Scenarios 5-1 and 5-2 appear to have a reverse pattern.

Now recall that both Scenarios 5-1 and 5-2 represent the events of premature default of the issuer when the CAT bond is not triggered and is triggered after the issuer defaults, respectively. Besides, as mentioned before, the higher the K^D value is, the more likely the issuer defaults earlier before time T. Therefore, the probabilities of these two scenarios increase with the issuer's bankruptcy level, and eventually dominate the change in the total price. For example, for the case $K^D = 100$, both scenarios contribute about 60% of the total price. This explains why the overall price increases even though the most scenarios present a decreasing trend as the value of K^D increases.

3. Recovery rate, β

   Figure 4.13 illustrates the relationship between $P_d(0,T)$ and the recovery rate, β. In the case of default, the CAT bond investors receive the recovery payment at the default time. We can see from the figure that the CAT bond price decreases sharply from about \$77.60 to \$48.63 as the value of β drops from 0.8 to 0.4. Owing to the possible decreasing recovery payoff, the CAT bond tends to be infavouered by the investors, and further affects the bond price.

Figure 4.13: CAT bond prices for three different values of β



$**$: base case.

Similarly as in 1., changing the value of β affects the payoff much more than the probability. The Trend column of Table 4.23 displays the changes in percentage, relative to the total price, for each scenario. The table shows that the total contribution of Scenarios 3, 4, 5-1 and 5-2 to the total CAT bond price rises from about 59% to 74% when the recovery rate, β, increases from 0.4 to 0.8. Thus, the total increasing percentage of these four scenarios significantly influences the overall change in the total CAT bond price.

Table 4.23: Percentages of total price for three different values of β

Scenario		Percentage of total price			
		$\beta = 0.4$	$\beta = 0.6^{**}$	$\beta = 0.8$	Trend
i.	1	37.9071%	29.1464%	23.7572%	↓
	2	0.0182%	0.0140%	0.0114%	↓
ii.	3	7.7706%	8.9621%	9.7399%	↑
	4	0.0067%	0.0078%	0.0084%	↑
iii.	5-1	15.2853%	17.6291%	19.1592%	↑
	5-2	35.9441%	41.4556%	45.0538%	↑
	6	3.0680%	2.7850%	2.2700%	↓

∗∗ : base case; ↑ : increasing trend; ↓ : decreasing trend.

Lastly, we conclude this section with a summary of the patterns for parameters a, K^D and β in Table 4.24. The CAT bond price increases with all of the three parameters.

Table 4.24: Impacts of parameters a, K^D and β on $P_d(0,T)$

Increase in parameter value	Effect on $P_d(0,T)$
a	↑
K^D	↑
β	↑

↑ : increasing trend; ↓ : decreasing trend.

Chapter 5

Conclusion

In this project, we have studied the CAT bond, one of the popular insurance-linked securities for insurance companies to transfer the catastrophe risk to the capital market. We provide a pricing formula for defaultable CAT bonds with liquidity risk under a stochastic interest rate process. In the pricing model, we also take into account the event of an issuer's premature default; so the payoffs may occur either prior to or at the maturity date, depending on when the issuer goes bankrupt. In addition, in order to keep up with an increasing trend in the number of future catastrophic events due to global warming, we assume that the issuer's losses are governed by a compound doubly stochastic Poisson process with intensity rate following a geometric Brownian motion instead of using a typical Poisson process with a constant intensity rate. This model allows one to investigate the relationships between the CAT bond price and interest rate risk, default risk, as well as liquidity risk.

The model is illustrated with numerical experiments and a sensitivity analysis. We first simulate the price of a default-free CAT bond, and compare the result to that of a defaultable CAT bond. It is not surprising that the incorporation of default risk reduces CAT bond prices since the investors have to bear the risk of not getting their full face value back at the end of the CAT bond term. Next, we show how the price of a defaultable CAT bond varies with key parameters, including the parameters of the interest rate model, asset dynamics, aggregate loss model, and liquidity process. We demonstrate the importance of taking liquidity risk and stochastic arrivals of catastrophic events into account. In addition, we examine the impacts of the ratio of the face value of a CAT bond to the issuer's total debts, the issuer's bankruptcy level, and the recovery rate on the CAT bond price. It is noteworthy that not all results follow the intuition; for example, adjust the issuer's bankruptcy to a lower level does not make the CAT bond more valuable.

For the future work, we can extend our pricing model to include moral hazard and basis risk studied in Lee and Yu (2002). Since the industry loss trigger is one of the commonly used types of triggers in the industry, an integrated analysis of moral hazard and basis risk would be an important step to deliver a more detailed scheme for the CAT bond's issuers to manage catastrophe risks.

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