ANALYSIS OF LONG-TERM DISABILITY INSURANCE PORTFOLIOS WITH STOCHASTIC INTEREST RATES AND MULTI-STATE TRANSITION MODELS

by

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Abstract

A general long-term disability insurance portfolio with semiannual disability payments and a lump sum death benefit payment is studied in this project. The transitions for policyholders in this portfolio, between the healthy, temporarily disabled, permanently disabled and the deceased statuses, are assumed to follow a continuous-time Markov process. The cash flow method is applied to study the first and second moments of the present value of future benefit payments and evaluate the total riskiness of the general insurance portfolio, which is decomposed into its insurance risk and investment risk. An alternative recursive method based on the term of the insurance policy is also demonstrated for the moment calculations of a single policy case. Two stochastic interest rate models, a binomial tree model and an AR(1) process, and a deterministic interest rate model are considered and illustrated.

Keywords: Long-term Disability Insurance Portfolio; Multi-state Transition Models; Binomial Tree Model; AR(1) process; Investment Risk; Insurance Risk

To my dearest family and my beloved grandpa who has passed away five months ago.

“Where there is a will, there is a way.”

— Thomas Edison

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

Most insurers offer separate insurance policies which provide financial supports to policyholders upon sickness, disability or death of the policyholder. The most common traditional life insurance products are the term life insurance policy, and the endowment. Another important type of insurance products are the long-term care annuity products including the disability insurance and the elder care insurance products, which are crucial to the social security system.

In the literature of life insurance and long-term care insurance, we have seen papers discussing the valuation of insurance policies or portfolios of such products. For example, Beekman (1990) presented a premium calculation procedure for long term care insurance by studying the random variable of first time loss of independence of Activities of Daily Living (ADL). The data used was based on the result of a survey to the non-institutionalized elderly people in Massachusetts in 1974. Parker (1997) introduced two cash flow approaches to evaluate the average risk per policyholder for a traditional term life and endowment insurance portfolio and decomposed the total riskiness into the insurance risk and the investment risk by conditioning on the survivorship and the interest rate, respectively. An Ornstein-Uhlenbeck process was applied to model the interest rate.

In general, the insurance risk (also called "the mortality risk") of an insurance portfolio results from mortality, disability or sickness. The average insurance risk per policy tends to zero as the number of contracts in the insurance portfolio goes to infinity under independent mortality assumption. Therefore, the insurance risk can be managed by risk pooling within the insurance company. The insurance risk for an insurance portfolio with large size is relatively small compared to the investment risk which comes from the fluctuations and the correlations of the periodic interest rates according to Parker (1997). Marceau and Gaillardetz (1999) presented the reserve calculation for a life insurance portfolio under stochastic mortalities and AR(1) interest rate assumptions.

However, it is not accurate to evaluate the insurance risk of the insurance company by separately evaluating the risk of each insurance portfolio consisting of only one type of insurance product. There might be some policyholders who are insured simultaneously by different types of insurance products (see Figure 1.1). Therefore, independent mortality assumption does not hold in these cases.

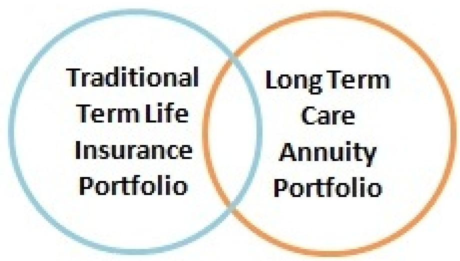

Figure 1.1: The Pools for ABC Insurance Company

To better evaluate and manage the insurance risk resulting from both the disability insurance product and the traditional life insurance product, we propose in this project a long-term disability insurance product which has features of both the death benefit payment and the disability annuity payments. The insurance risk of such portfolios is expected to be lower than that of a term life insurance portfolio plus that of a disability annuity portfolio since the annuity payments will partially offset the insurance risk resulting from mortality.

For the insurance model we study, there are four statuses for the insureds within this long-term disability insurance product: Healthy, Temporarily Disabled, Permanently Disabled and Deceased. For this disability model, we allow the possibility of recovering from temporarily disabled but permanently disabled implies no possibility of any recovery. The policyholder who is insured by this product shall be paid the semiannual disability benefit during the period of disability and a lump sum death benefit upon the occurrence of the death. Considering such an insurance portfolio, two key issues need to be addressed for valuation purposes, namely selecting an interest rate model and a methodology for calculating the transition probabilities including the assumptions of the forces of transition.

An appropriate interest rate model is a significant factor to accurately estimate the investment risk of an insurance portfolio. The deterministic interest rate model can be seen as the simplest interest rate model. In practice, this model is a common choice for the short-term valuation. In other words, the actuaries usually predetermine the annual interest rate for a short period (normally 3-5 years) and then adjust this rate according to the market environment after the valuation period. In spite of its simplicity, the disadvantage of this deterministic model is that it does not allow any fluctuations on the interest rate within the valuation period and thus does not reflect the market variation promptly.

Since 1970s, several stochastic models have been studied to provide the flexibility in modeling the interest rates. Panjer and Bellhouse (1980) introduced the application of the autoregressive processes of order 1 and 2 (special cases of AR(p) process) and their continuous time analogues in the life insurance context. They argued that a more general autoregressive moving average process (ARMA process) can also be applied to model the interest rate. In addition, Cairns (2004) illustrated several other interest rate models such as the Vasicek model, the discrete time binomial model and the Cox-Ingersoll-Ross (CIR) model. Gaillardetz (2007) analyzed the impact of the interest guarantee on the future liabilities with a binomial tree interest rate model by assuming time constant annual volatilities. The binomial tree interest rate model is a good choice for long-term valuation purpose when the future market condition is unknown and the long-term mean of the one-period interest rate might move. The parameters of such model can be estimated by observing the average historical market volatility.

Ideally, an appropriate interest rate model should be estimated using real data. Several papers have modeled the interest rate from the data. For example, Giaccotto (1986) developed a methodology for moment calculations of insurance functions with both the stationary interest rate process and non-stationary process such as the autoregressive integrated moving average process (ARIMA). The Vasicek model used for zero-coupon bond pricing was also illustrated to obtain the present value of the life insurance functions. The Durand one-year spot rate, the prime commercial paper and the medium grade preferred stock data were used to test the stationarity of the interest rate model. Parker (1997) applied the Ornstein-Uhlenbeck process as a model for the instantaneous rate of return in order to discount the expected insurance cash flows. The parameters are estimated from past data which reflects the investment strategy of the insurer with respect to the insurance portfolio.

Instead of estimating and selecting an interest rate model from data, in this project, we directly consider three interest rate models: the deterministic model, the binomial tree model and the AR(1) model. That lets us focus on the methodology of the risk analysis of the long-term disability insurance portfolio. Hence, we just set up the values of the parameters of three interest rate models applied and we shall do some sensitivity tests of the parameters chosen.

As we mentioned earlier, modeling the transition process of the statuses of the policyholders is another key issue which has a huge impact on the valuation results. A few papers discussing the health insurance cases have been published since 1980s. Waters (1984) gave the basic concepts of the transition probabilities, the forces of transition and their relationships. Ramsay (1984) studied the ruin probability of the surplus of a sickness insurance contract which pays the benefit only if the duration of the sickness exceeds certain period. Waters (1990) illustrated a method of calculating the moment of benefit payments for a sickness insurance contract by introducing the semi-Markov chain to model the transition process. In Jones (1994), a Markov chain model was presented to calculate the transition matrix of a multi-state insurance contracts consisting of three states with one-direction transitions only (e.g. healthy, permanently disabled and deceased). Constant and piecewise constant forces of transition were assumed in the paper. Levikson and Mizrahi (1994) priced a long term care contract with three different care levels assuming that the policyholder's health status could only either remain unchanged or deteriorated.

Haberman et al.∼ (1997) introduced a general multi-state model for different types of long term care (LTC) insurances including

1. stand-alone policy, which provides fixed annuity amount for policyholders at different frailty levels;

2. enhanced annuity, which provides annuity payments to current residents of nursing care homes;

3. LTC cover as a rider benefit, which can be regarded as a pre-death LTC payment;

4. enhanced pension, which combines both a standard post-retirement pension annuity and an extra LTC payment; and

5. insurance packages, which consists of LTC benefit payment, deferred life annuity payment and a lump sum death benefit payment.

This paper concentrates on the calculations of expected present values of payment streams. Numerical results are given under both the time-continuous Markov chain process transition assumption and the discrete approach. Haberman et al.∼ (1997) also mentioned another popular transition process applied in valuations of insurance policies, the semi-Markov model, which allows the effective modeling of duration dependence. The semi-Markov chain model could be appropriately used when large sizes of insurance data is available and it would provide more accuracy than the Markov chain model by considering the dependency between different statuses of the policyholders. This process can be achieved by getting the distribution matrix for duration of each specific transition as well as the proportions of each specific transition within certain time periods. In their paper, the Danish insurance data was used to estimate the piecewise constant forces of transition. The deterministic interest rate model was applied for valuation purposes. However, recoveries from higher frailty levels were not considered in Haberman et al.∼ (1997). More details on this topic can be found in Haberman and Pitacco (1998).

In this project, we study a four-state model with two-directions transitions for a long-term disability insurance under the piecewise constant forces of transition assumption. We calculate the transition probabilities by establishing the relationships among the forces of transition using the survey results illustrated in Rajnes (2010) and the assumptions in Haberman et al.∼ (1997). The methodology is similar to the discrete-time Markov chain approach illustrated in Haberman et al.∼ (1997) except that we consider the situation of recoveries from higher frailty levels and we check the statuses of the policyholders semi-annually for the insurance model we study. The service table in Bowers et al.∼ (1997) is applied to calculate the transition probabilities of the policyholders at work. Kolmogorov's forward equations introduced in Daniel (2004) and Dickson et al.∼ (2009) are shown as an alternative way to calculate the transition probabilities for our four-state disability insurance model. Our work can be seen as an extension to the results in Parker (1997). We calculate the moment of the future liabilities and analyze the risk of a long-term disability insurance portfolio which provides both the disability annuity payments and a lump sum death payment with the possibility of recoveries. An alternative recursive method based on the benefit payment variable is also derived to calculate the moments of the future liabilities for a single long-term disability insurance policy. Note that the dependent mortality structure is not considered in this project.

The rest parts of this thesis are organized as follows. Chapter 2 introduces the basic concepts and framework of the multi-state transition model under a Markov chain process assumption. Four practical examples and their transition probabilities are presented. A long-term disability insurance model is introduced with details, that is the model we study in this thesis. Chapter 3 discusses three interest rate models: the deterministic interest rate model, the binomial tree model and the AR(1) model. Formulas are derived for some basic interest rate functions, such as the mean, variance and the auto-covariance of the discounted interest rate process. The mortality assumptions for the transition intensities are also discussed. In Chapter 4, a recursive formula and its alternative cash flow method are presented for calculating the moments of the future liabilities in the single policy case. In Chapter 5, homogeneous and non-homogeneous long-term disability insurance portfolios are considered, respectively. Closed-form formulas for calculating the average insurance and investment risks for portfolio cases are provided. In Chapter 6, numerical illustrations are shown for the general insurance portfolios described in the previous chapters with independent transitions assumption and three interest rate models: deterministic, binomial tree model and the AR(1) process. Chapter 7 concludes the project and discusses the future work that could be done.

Chapter 2

The Discrete Time Multi-State Transition Model

In this chapter, we introduce a multi-state transition model under the Markovian assumption, which is widely applied in the insurance industry. The definitions and the actuarial notations of the multi-state transition model and the non-homogeneous Markov chain are given in Section 2.1. Several practical examples together with their transition matrices are provided in Section 2.2. Section 2.3 presents a long-term disability income model studied in this project.

2.1 Concepts of the Multi-State Transition Model

Multi – state transition models are probability models that describe the random movements of a subject among various states. Daniel (2004) presented such models with applications in Actuarial Science. For practical interest of the survival and failure rates, the subject could be either a person or a piece of machinery or even a loan contract.

To describe the transition process of a subject moving among different states, a commonly used model is the Markov chain model. Now we first define the state space, the Markov chain process and its transition probabilities in a life insurance context.

Definition 2.1 For a policyholder who joins an insurance plan at age x, assume there are r different states such as healthy, disabled or deceased at year t,

1. Define the finite integer-valued set $\Xi = \{0, 1, \ldots, r-1\}$ as the **state space** of the

Markov chain model;

2. Define $M_t(x) \in \Xi$ as the state in which the policyholder is at age $x+t$;

3. If the state of the policyholder at time $t+s$ does not depend on the states prior to time t for $x \in \mathbb{N}^+, t, s > 0$, then the state process $\{M_t(x), t \geq 0\}$ is called a **continuous-time Markov chain process**. Mathematically, this means

$$Pr[M_{t+s}(x) = j \mid M_t(x) = i] = Pr[M_{t+s}(x) = j \mid M_t(x) = i, M_u(x) = s(x,u), 0 \leq u < t]$$

for $t \geq 0$, where $s(x,u) \in \Xi$ refers to the status at age $x+u$. The **transition probability** of this Markov chain process is denoted by

$$_sp^{ij}(x+t) = Pr[M_{t+s}(x) = j \mid M_t(x) = i], \qquad x \in \mathbb{N}^+, t, s \geq 0, i, j \in \Xi.$$

Since for life insurance applications, the transition process among different states are age dependent, we thus only focus on the non-homogeneous Markov chain process. In other words, the transition probability stated above depends on both the duration s and the age $x+t$. However, a special case is the homogeneous Markov chain process where $x+t$ has no impact on the transition probability.

Definition 2.2 Suppose that we check the status of the policyholder every t_0 year(s), say, at times $t_0, 2t_0, \ldots$. The corresponding states of the continuous-time Markov process at the observation times are denoted by $M_{it_0}(x)$ for $t_0 > 0$ and $i \in \mathbb{N}$. Then denote $Q_{(x,k,t_0)}$ the transition matrix which stands for the transition probability after k observation periods with equal length t_0 year(s) given the policyholder's status at age x. Its ij-th element is then defined as follows:

$$Q^{ij}_{(x,k,t_0)} = Pr[M_{kt_0}(x) = j \mid M_0(x) = i], \qquad k, x \in \mathbb{N}^+, t_0 > 0, i, j \in \Xi. \tag{2.1}$$

Furthermore, a general transition matrix property is given as:

$$Q_{(x,k,t_0)} = Q_{(x,l,t_0)} \times Q_{(x+t_0 l, k-l, t_0)}, \qquad 0 < l \leq k, l, k, x \in \mathbb{N}^+, t_0 > 0, i, j \in \Xi. \tag{2.2}$$

2.2 Practical Examples of The Multi-State Model

In this section, we give some special examples of the multi-state transition model in the actuarial context to demonstrate its framework under the Markov chain technique. Some of these examples can also be found in Daniel (2004); see also Dickson et al (2009). Note that notations, such as $p_{x+t}\,, q_{x+t}\,, q^{(i)}_{x+t}$ and ${}_k p_{x+t,y+t}$, are standard actuarial notations in Actuarial Mathematics; see Bowers et al.~ (1997) for more details.

Figure 2.1: Basic Survival Model

Example 2.1. (Basic Survival Model) In the basic survival model (see Figure 2.1), we concentrate on two states of a subject: intact and failed. This basic model describes a subject moving from the survival state (state 0) to the absorbing failure state (state 1). For instance, if the subject is a policyholder who is insured at age x, then the closed form for the k-period transition matrix can be easily obtained as:

$$Q_{(x,k,t_0)} = \prod_{i=0}^{k-1} Q_{(x+t_0 i,1,t_0)} = \begin{bmatrix} \prod_{i=0}^{k-1} {}_{t_0}p_{x+t_0 i} & 1-\prod_{i=0}^{k-1} {}_{t_0}p_{x+t_0 i} \\ 0 & 1 \end{bmatrix}, \qquad k\,,x \in \mathbb{N}\,, t_0 \geq 0\,.$$

This model could also be applied to examine the failure rate of a machinery.

Example 2.2. (Multiple-decrement Survival Model) Unlike the basic survival model, this model has classified the failed state into several sub-states corresponding to the causes that give rise to the failure of the subject such as heart disease, car accident, or murder etc (see Figure 2.2). As illustrated by Figure 2.2, for this model, state 0 stands for the survival status and state i refers to failure owing to reason i for $i = 1, 2, \ldots, n$. The one-period

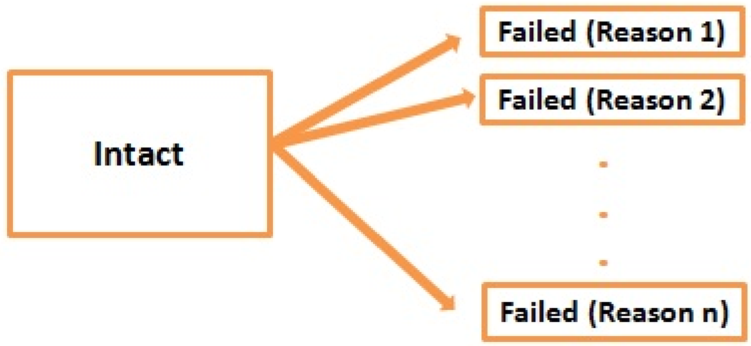

Figure 2.2: Multiple-Decrement Survival Model

transition matrix of this $(n+1)$-state multiple-decrement survival model is given by

$$Q_{(x,1,t_0)} = \begin{bmatrix} {}_{t_0}p_x & {}_{t_0}q_x^{(1)} & \dots & \dots & \dots & {}_{t_0}q_x^{(n)} \\ 0 & 1 & \dots & \dots & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \qquad x \in \mathbb{N}\,, t_0 > 0\,, \tag{2.3}$$

where the first row of the matrix stands for the probabilities of survival and failure owing to reasons $1, 2, \dots, n$, respectively, with ${}_{t_0}p_x + \sum_{i=1}^{n} {}_{t_0}q_x^{(i)} = 1$. Similarly to Example 2.1, the general expression for the k-period transition matrix $(k > 1)$ are given in the following proposition.

Proposition 2.1. For $k\,, x \in \mathbb{N}$ and $t \geq 0$,

$$Q_{(x,k,t_0)} = \prod_{i=0}^{k-1} Q_{(x+t_0 i,1,t_0)} = \begin{bmatrix} \prod_{i=0}^{k-1} {}_{t_0}p_{x+t_0 i} & 0 & \dots & \dots & 0 \\ \sum_{j=0}^{k-2} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(1)}_{x+t_0(j+1)} \right] & 1 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \sum_{j=0}^{k-2} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(n)}_{x+t_0(j+1)} \right] & 0 & \dots & 0 & 1 \end{bmatrix}^T . \tag{2.4}$$

Proof. The proof is by induction on period k. Obviously, (2.4) holds when $k = 1$. Assume the formula holds for any $Q_{(x,k-1,t_0)}$ for $k\,, x \in \mathbb{N}^+\,, k > 1$ and $t_0 > 0$, which is

$$Q_{(x,k-1,t_0)} = \begin{bmatrix} \prod_{i=0}^{k-2} {}_{t_0}p_{x+t_0 i} & 0 & \dots & \dots & 0 \\ \sum_{j=0}^{k-3} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(1)}_{x+t_0(j+1)} \right] & 1 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \sum_{j=0}^{k-3} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(n)}_{x+t_0(j+1)} \right] & 0 & \dots & 0 & 1 \end{bmatrix}^T .$$

Then for k-period, by (2.2),

$$
\begin{aligned}
Q_{(x,k,t_0)} &= Q_{(x,k-1,t_0)} \times Q_{(x+t_0(k-1),1,t_0)} \\
&= \begin{bmatrix}
\prod_{i=0}^{k-2} {}_{t_0}p_{x+t_0 i} & 0 & \dots & \dots & 0 \\
\sum_{j=0}^{k-3} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(1)}_{x+t_0(j+1)} \right] & 1 & & & \vdots \\
\vdots & 0 & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & 0 \\
\sum_{j=0}^{k-3} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(n)}_{x+t_0(j+1)} \right] & 0 & \dots & 0 & 1
\end{bmatrix}^T \\
&\times \begin{bmatrix}
{}_{t_0}p_{x+t_0(k-1)} & 0 & \dots & \dots & \dots & 0 \\
{}_{t_0}q^{(1)}_{x+t_0(k-1)} & 1 & 0 & \dots & \dots & 0 \\
\vdots & & \ddots & & & \vdots \\
\vdots & & & 1 & & \vdots \\
\vdots & & & & \ddots & \vdots \\
{}_{t_0}q^{(n)}_{x+t_0(k-1)} & 0 & \dots & \dots & \dots & 1
\end{bmatrix}^T \\
&= \begin{bmatrix}
\prod_{i=0}^{k-1} {}_{t_0}p_{x+t+t_0 i} & 0 & \dots & \dots & 0 \\
\sum_{j=0}^{k-2} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t_0 i} \right) {}_{t_0}q^{(1)}_{x+t_0(j+1)} \right] & 1 & & & \vdots \\
\vdots & 0 & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & 0 \\
\sum_{j=0}^{k-2} \left[\left(\prod_{i=0}^{j} {}_{t_0}p_{x+t+t_0 i} \right) q^{(n)}_{x+t_0(j+1)} \right] & 0 & \dots & 0 & 1
\end{bmatrix}^T , \qquad k\,, x \in \mathbb{N}^+\,, k > 1\,, t_0 > 0\,.
\end{aligned}
$$

This proves that (2.4) is true for all $k \in \mathbb{N}^+$. □

Example 2.3. (Joint Life Model) This model has been widely applied in the pension industry to describe the joint and last survival status of couples. Define state 0 as both husband and wife alive, state 1 as husband alive and wife dead, state 2 as husband dead and wife alive, and the absorbing state 3 as both husband and wife dead (see Figure 2.3). In this example, we need two age variables, x and y, to describe respectively the entry ages of the husband and wife. We use the notation $Q_{((x,y),k,t_0)}$ for the k-period transition matrix in this case. It is not difficult to obtain a general formula of the k-period transition matrix for this multiple life model:

$$Q_{((x,y),k,t_0)} = \prod_{i=0}^{k-1} Q_{((x+t_0 i, y+t_0 i),1,t_0)}$$

$$= \begin{bmatrix} {}_{t_0k}p_{xy} & {}_{t_0k}p_x - {}_{t_0k}p_{xy} & {}_{t_0k}p_y - {}_{t_0k}p_{xy} & 1 - {}_{t_0k}p_x - {}_{t_0k}p_y + {}_{t_0k}p_{xy} \\ 0 & {}_{t_0k}p_x & 0 & {}_{t_0k}q_x \\ 0 & 0 & {}_{t_0k}p_y & {}_{t_0k}q_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.5)$$

for $k\,, x\,, y \in \mathbb{N}$ and $t_0 > 0$.

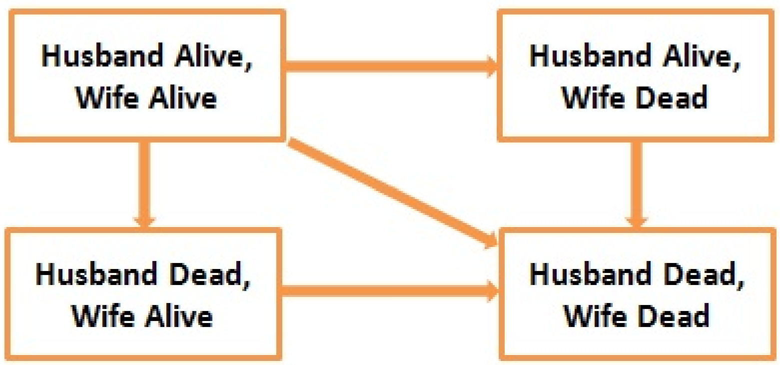

Figure 2.3: Joint Life Model

Note that (2.5) is applicable under both the independent and dependent mortality assumptions for joint couples, such as copula and common shock models discussed in Chen (2010). Under the independent mortality assumption, we have ${}_kp_{xy} = {}_kp_x \times {}_kp_y$. Thus, the transition matrix (2.5) for this special case can be further simplified to

$$Q_{((x,y),k,t_0)} = \begin{bmatrix} {}_{t_0k}p_x \times {}_{t_0k}p_y & {}_{t_0k}p_x \times {}_{t_0k}q_y & {}_{t_0k}q_x \times {}_{t_0k}p_y & {}_{t_0k}q_x \times {}_{t_0k}q_y \\ 0 & {}_{t_0k}p_x & 0 & {}_{t_0k}q_x \\ 0 & 0 & {}_{t_0k}p_y & {}_{t_0k}q_y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $k\,, x\,, y \in \mathbb{N}$ and $t_0 > 0$.

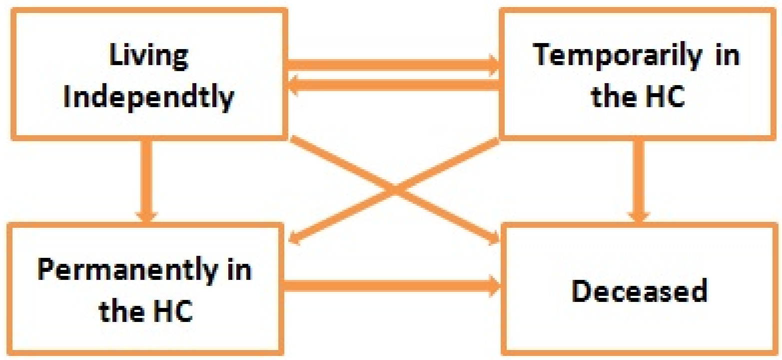

Figure 2.4: The Continuing Care Retirement Communities Model

Example 2.4. (CCRC's) The Continuing Care Retirement Communities model have four states (see Figure 2.4). State 0 refers to a resident living independently, state 1 stands for temporarily living in the Health Center, state 2 stands for permanently living in the Health Center, while the absorbing state 3 stands for deceased. Thus, ${}_{t_0k}p^{3j}(x) = 0$ for $j = 0, 1, 2$ and ${}_{t_0k}p^{33}(x) = 1$ for $k \in \mathbb{N}^+$.

After looking through the four examples above, we could classify the multi-state transition model into two types: the models in Examples 2.1 - 2.3 do not allow two-way transitions and thus are called **one-way multi-state transition model**; whereas Example 2.4 allows recovery for the policyholders and thus is called **two-way multi-state transition model**. In Section 2.3, we introduce another example of two-way multi-state transition model.

2.3 The Long-Term Disability Insurance Model

The model introduced in this section is the main model studied in this project; the framework of which is very similar to Example 2.4, except that the long-term disability model describes the disability status instead of the living dependency status. Overall, there are four states for a certain x-year-old policyholder in the model. State 0 refers to healthy, state 1 refers to temporarily disabled, state 2 refers to permanently disabled and state 3 stands for deceased (see Figure 2.5). Once a policyholder becomes permanently disabled, he or she cannot

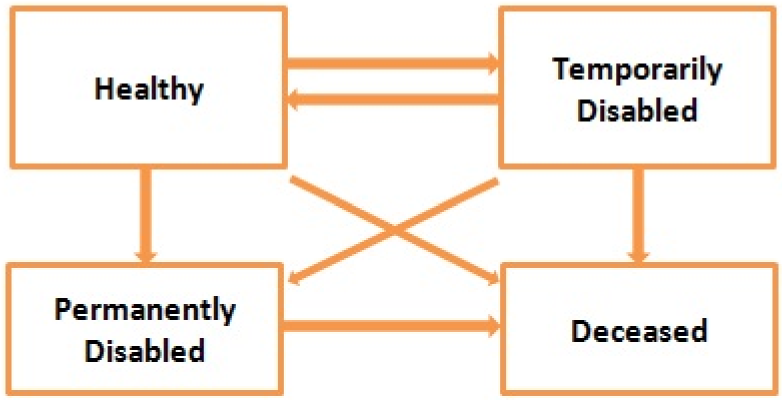

Figure 2.5: The Long-term Disability Insurance Model

recover from the disability anymore. However, chances are that a policyholder recovers from a temporary disability.

In addition, for simplicity of valuation purposes, we assume

1. there is no pending time from the moment of turning disabled or deceased for an insured to the moment that the insurer receives the relevant proof documents;

2. all the policyholders are healthy at the effective date of the insurance policy. In other words, we have $M_0(x) = 0$, where x stands for the entry age of the policyholder and $x \in \mathbb{N}^+$.

If a status check is done every t_0 year(s), the general formula of the k-period transition matrix is denoted by

$$Q^{ij}_{(x,k,t_0)} = Pr\left[M_{t_0 k}(x) = j \,|\, M_0(x) = i\right], \qquad k, x \in \mathbb{N}^+, t_0 > 0, i, j \in \Xi. \tag{2.6}$$

In this thesis, we design a long-term disability insurance product for which the status check is done semiannually. In other words, $t_0 = 0.5$. The reason for selecting this semiannual frequency of checking the states shall be explained in Chapter 5. Therefore, hereby we have a special case for the k-period transition matrix:

$$Q^{ij}_{(x,k,0.5)} = Pr\left[M_{0.5k}(x) = j \,|\, M_0(x) = i\right], \qquad k, x \in \mathbb{N}^+, i, j \in \Xi. \tag{2.7}$$

According to the transition rule of the model, the k-period transition matrix for the disability income model for an x-year-old policyholder is of the form:

$$Q_{(x,k,0.5)} = \begin{bmatrix} Q^{00}_{(x,k,0.5)} & Q^{01}_{(x,k,0.5)} & Q^{02}_{(x,k,0.5)} & Q^{03}_{(x,k,0.5)} \\ Q^{10}_{(x,k,0.5)} & Q^{11}_{(x,k,0.5)} & Q^{12}_{(x,k,0.5)} & Q^{13}_{(x,k,0.5)} \\ 0 & 0 & Q^{22}_{(x,k,0.5)} & Q^{23}_{(x,k,0.5)} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad k\,, x \in \mathbb{N}^{+}\,,$$

where the explicit formulas for these non-zero transition probabilities are provided in Chapter 4. The detailed transition assumptions and the methodology such as the Kolmogorov's equation shall be introduced in Chapters 3 and 4.

Chapter 3

Model Assumptions

3.1 Interest Rate Models

3.1.1 Summary and Actuarial Notations

For actuarial valuation purposes, the interest rate assumption is a significant element for premium and reserve calculations. In this chapter, we introduce several interest rate models applied in our valuations including the deterministic interest rate model, the binomial tree model and the autoregressive model of order 1, the AR(1) model.

Define δ_i as the effective annual interest rate over the time interval $[i, i{+}1)$ year(s) and assume the starting value δ_0 is known. Hereby we define the discounted interest rate function over the time interval $[0, s)$ years as follows:

$$V(\delta_0, s) = \begin{cases} \left[\prod_{i=0}^{[s]-1} (1+\delta_i)^{-1}\right] (1+\delta_{[s]})^{-\{s\}}\,, & s > 1 \\ (1+\delta_0)^{-s}\,, & 0 \leq s \leq 1 \end{cases}, \tag{3.1}$$

where $[s]$ is the integer part of s, and $\{s\}$ is the decimal part of s. In this project, each interest rate period is a policy year.

In the following sections, we show the framework of three different interest rate models and derive their interest rate functions in detail.

3.1.2 Deterministic Interest Rate Model

In this section, we introduce two types of deterministic interest rate models: one with single scenario and the other with multiple scenarios (Bowers et al.$\sim$ 1997). In this project, we apply the former model, which sets the annual interest rates δ_i to be constant over time for simplicity purpose:

$$E[V(\delta_0, s)] = V(\delta_0, s) = \left[\prod_{i=0}^{[s]-1} (1+\delta_i)^{-1}\right] (1+\delta_{[s]})^{-\{s\}} = (1+\delta_0)^{-s}\,, \qquad s \geq 0\,. \quad (3.2)$$

On the contrary, the latter model is also a popular choice applied in the industry: a sequence of annual interest rates, $\{\delta_0, \delta_1, \ldots,\}$, for several years (normally 3-5 years) are predetermined by an actuarial perspective of the future economic environment.

3.1.3 Binomial Tree Model

In this section, we introduce a frequently used model which is used to describe the price movement of financial derivatives and stocks, the binomial tree model. Assume that the annual interest rate each year is an independent trial, starting from the value of $r(0,0)$ in the first period. The annual interest rate for the next year moves up to $r(1,1)$ with probability p and moves down to $r(1,0)$ with probability $1-p$ (see Figure 3.1).

Once the starting value of the annual interest rate is given, the annual interest rate for the $[k, k+1)$ period, δ_k, follows a binomial distribution with parameters k and p with probability

$$Pr\left[\delta_k = r(k,j)\right] = p^j (1-p)^{k-j}\,, \qquad j = 0, 1, \ldots, k\,;\, k \in \mathbb{N}\,, \quad (3.3)$$

where $r(k,j)$ stands for j ups during the time interval $[0, k+1)$ years.

Similar to Gaillardetz (2007), we set the probabilities of moving up and down to be equal (i.e. , $p = 0.5$). Now we assume the annual volatility, $\{\sigma_r(k)\}$, which can be analyzed from the capital market environment and has a relationship with $r(k,j)$, as is described in the following proposition.

Proposition 3.1. (Black et al.~ 1990) For $j = 0, 1, \ldots, k-1\,;\, k \in \mathbb{N}^+$, the annual volatility for the time period $[k-1, k)$ is defined as

$$[\sigma_r(k)]^2 = Var[\,ln(\delta_k)\,|\,\delta_{k-1} = r(k-1, j)]\,,$$

and its relationship with the annual interest rate can be expressed as:

$$[\sigma_r(k)]^2 = \left\{0.5\,ln\left[\frac{r(k, j+1)}{r(k,j)}\right]\right\}^2 . \quad (3.4)$$

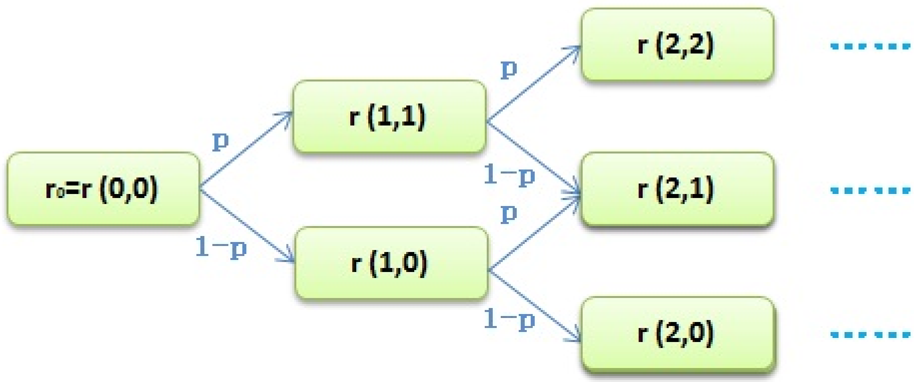

Figure 3.1: The Binomial Tree Model

Proof. Since

$$ln(\delta_k)\,|\,[\delta_{k-1}=r(k-1,j)] = \begin{cases} ln\,[r(k,j+1)] & \text{with probability } 0.5 \\ ln\,[r(k,j)] & \text{with probability } 0.5 \end{cases},$$

thus the first and second moments of the annual interest rate for the period $[k, k+1)$ are as follows:

$$E[ln(\delta_k)\,|\,\delta_{k-1}=r(k-1,j)] = 0.5\,\{ln\,[r(k,j)] + ln\,[r(k,j+1)]\}\ ,$$
$$E[ln^2(\delta_k)\,|\,\delta_{k-1}=r(k-1,j)] = 0.5\,\left\{ln^2\,[r(k,j)] + ln^2\,[r(k,j+1)]\right\}\ .$$

Therefore, the variance of the annual interest rate for $[k, k+1)$ can be derived as:

$$\begin{aligned} &Var[ln(\,\delta_k)\,|\,\delta_{k-1}=r(k-1,j)] \\ =&0.25\,\{ln\,[r(k,j+1)] - ln\,[r(k,j)]\}^2 \\ =&\left\{0.5\,ln\left[\frac{r(k,j+1)}{r(k,j)}\right]\right\}^2, \qquad j=0,1,\ldots,k-1\,;\; k\in\mathbb{N}^+\,. \end{aligned} \tag{3.5}$$

□

Now assume the annual volatility $\sigma_r(k)$ is constant over time, then by Proposition 3.1, we can further obtain a relationship between the values of two adjacent interest rates, $r(k, j)$ and $r(k, j-1)$, at any time k:

$$\frac{r(k,j+1)}{r(k,j)} = e^{2\sigma_r(k)}\,, \qquad j = 0,1,\ldots,k-1\,;\; k \in \mathbb{N}^+\,. \tag{3.6}$$

In addition, denote $L(0,T)$ the current price of the zero-coupon bond which has a maturity of T years and its value is estimated with the risk-free interest rate r.

Given the starting value of the annual interest rate, there are k possible values for δ_k with 2^k possible paths. Therefore, the expectation of $V(\delta_0,k)$, the discounted interest rate factor over $[0,k)$ years for $k \in \mathbb{N}^+$ can be calculated as follows by matching the market price and the annual interest rate in the binomial tree model (see Gaillardetz (2007)):

$$\begin{aligned} E[V(\delta_0,k)] = L(0,k) &= \left(\frac{1}{1+r}\right)^k \\ &= \left(\frac{1}{2}\right)^{k-1} \frac{1}{1+\delta_0} \sum_{j_1=0}^{1} \sum_{j_2=j_1}^{j_1+1} \cdots \sum_{j_{k-1}=j_{k-2}}^{j_{k-2}+1} \left\{ \prod_{m=1}^{k-1} [1+r(m,j_m)]^{-1} \right\}. \end{aligned} \tag{3.7}$$

for $k > 1\,, k \in \mathbb{N}^+$. Rewriting (3.7), we have:

$$\frac{2^{k-1}(1+\delta_0)}{(1+r)^k} = \sum_{j_1=0}^{1} \sum_{j_2=j_1}^{j_1+1} \cdots \sum_{j_{k-1}=j_{k-2}}^{j_{k-2}+1} \left\{ \prod_{m=1}^{k-1} [1+r\,(m,j_m)]^{-1} \right\}. \tag{3.8}$$

By solving (3.6) and (3.8), we can obtain the values for all the annual interest rates $r(k,j)$. Since the status check is done semiannually for the disability insurance product studied in this project, k is not necessarily an integer. A more general formula of (3.7) can be derived as follows:

$$E[V(\delta_0,s)] = \begin{cases} \left(\frac{1}{1+\delta_0}\right)^s, & 0 \le s \le 1 \\ \frac{(\frac{1}{2})^{[s]-1}}{1+\delta_0} \sum_{j_1=0}^{1} \cdots \sum_{j_{[s]-1}=j_{[s]-2}}^{j_{[s]}} \left\{ \left[\prod_{m=0}^{[s]-1} \frac{1}{1+r(m,j_m)} \right] \left[\frac{1}{1+r\big([s],j_{[s]}\big)} \right]^{\{s\}} \right\}, & s > 1 \end{cases}. \tag{3.9}$$

The longest term of the insurance product we study is 25 years, which implies that there are 2^{25} paths of interest rate scenarios under the binomial tree model. Instead of using the time-consuming method provided by (3.9) to calculate the exact expectation of $V(\delta_0,k)$, we use random sampling method to approximate the expectation for coding purpose. For each trial, we generate a series of Bernoulli random variables with length of 25 to denote the up and down status and repeat this process for $1,000,000$ times.

If $s = u$, we have

$$E[V(\delta_0,s)V(\delta_0,u)] = E[V(\delta_0,s)^2]$$
$$= \begin{cases} (\frac{1}{1+\delta_0})^{2s}\,, & 0\leq s \leq 1 \\ \frac{(\frac{1}{2})^{[s]-1}}{(1+\delta_0)^2}\sum_{j_1=0}^{1}\cdots\sum_{j_{[s]-1}=j_{[s]-2}}^{j_{[s]}}\left\{\left[\frac{1}{1+r\big([s],j_{[s]}\big)}\right]^{2\{s\}}\left\{\prod_{m=0}^{[s]-1}\left[\frac{1}{1+r(m,j_m)}\right]^2\right\}\right\}, & s>1 \end{cases} \tag{3.10}$$

If $s < u$, we have

$$E[V(\delta_0,s)V(\delta_0,u)]$$
$$= \begin{cases} (\frac{1}{1+\delta_0})^{s+u}\,, & \mathbb{A} \\ \frac{(\frac{1}{2})^{[u]-1}}{(1+\delta_0)^2}\sum_{j_1=0}^{1}\cdots\sum_{j_{[u]-1}=j_{[u]-2}}^{j_{[u]}}\left\{\prod_{l=s\,,u}\left\{\left[\frac{1}{1+r\big([l],j_{[l]}\big)}\right]^{\{l\}}\left[\prod_{m=0}^{[l]-1}\frac{1}{1+r(m,j_m)}\right]\right\}\right\}, & \mathbb{B} \\ \frac{(\frac{1}{2})^{[u]-1}}{(1+\delta_0)^{s+1}}\sum_{j_1=0}^{1}\cdots\sum_{j_{[u]-1}=j_{[u]-2}}^{j_{[u]}}\left\{\left[\frac{1}{1+r([u],j_{[u]})}^{\{u\}}\right]\left[\prod_{m=0}^{[u]-1}\frac{1}{1+r(m,j_m)}\right]\right\}, & \mathbb{C} \end{cases} \tag{3.11}$$

where $\mathbb{A}$ stands for the situation that $0 \leq s\,,u \leq 1$, $\mathbb{B}$ stands for the situation that $s\,,u \geq 1$ and $\mathbb{C}$ stands for other situations.

3.1.4 AR(1) Model

In this section, we shall construct a stationary AR(1) process to model the interest rate. The key results can be found in Pandit and Wu (1983). Assume the start value of the force of interest for the time period $[0,1)$, $ln(1+\delta_0)$, is known and it is also the long-term mean of the annual force of interest, $ln(1+\delta_k)$:

$$ln(1+\delta_k) - ln(1+\delta_0) = \phi\,[ln(1+\delta_{k-1}) - ln(1+\delta_0)] + a_k\,, \qquad k \in \mathbb{N}^+\,, \tag{3.12}$$

where the residual $a_k \sim N(0, \sigma_a^2)$ are i.i.d. Gaussian random variables. By induction, we have

$$\begin{aligned} ln(1+\delta_k) - ln(1+\delta_0) &= \phi\,[ln(1+\delta_{k-1}) - ln(1+\delta_0)] + a_k \\ &= \phi\,\{\phi\,[ln(1+\delta_{k-2}) - ln(1+\delta_0)]\} + \phi a_{k-1} + a_k \\ &= \dots \\ &= \sum_{j=0}^{k-1} \phi^j a_{k-j}\,, \qquad k \in \mathbb{N}^+\,. \end{aligned} \tag{3.13}$$

The long-term mean and variance of the process can be expressed as:

$$E[ln(1+\delta_k)\,|\,\delta_0] = ln(1+\delta_0)\,, \qquad k \in \mathbb{N}\,, \tag{3.14}$$

and

$$\lim_{k\to\infty} Var[ln(1+\delta_k)|\,\delta_0] = \lim_{k\to\infty} Var\left[\sum_{j=0}^{k-1} \phi^j a_{k-j}\right] = \lim_{k\to\infty}\left[\sigma_a^2 \frac{1-\phi^{2k}}{1-\phi^2}\right] = \frac{\sigma_a^2}{1-\phi^2} \tag{3.15}$$

if the process is stationary (i.e. $|\phi| < 1$). In general, we can obtain the variance of the force of interest by applying (3.13):

$$Var[ln(1+\delta_k)\,|\,\delta_0] = Var\left[\sum_{j=0}^{k-1} \phi^j a_{k-j}\right] = \sum_{j=0}^{k-1} \phi^{2j} Var[a_{k-j}] = \frac{1-\phi^{2k}}{1-\phi^2}\sigma_a^2\,, \qquad k \in \mathbb{N}^+\,. \tag{3.16}$$

Proposition 3.2 Provided with n periods of past data for the annual interest rate prior to time 0, $\delta_{-n}, \delta_{-(n-1)}, \dots, \delta_{-1}$, we can apply the ordinary least square method to get the estimate of ϕ as follows:

$$\hat{\phi} = \frac{\sum\limits_{k=-(n-1)}^{-1} [ln(1+\delta_k) - ln(1+\delta_0)]\,[ln(1+\delta_{k-1}) - ln(1+\delta_0)]}{\sum\limits_{k=-(n-1)}^{-1} [ln(1+\delta_k) - ln(1+\delta_0)]^2}\,, \qquad k \in \mathbb{Z}\,. \tag{3.17}$$

The last step is to get the expected value and the auto-covariance of the discounted interest rates. Firstly, since the sequence $ln(1+\delta_s)$ has a normal pattern, the expectation of the discounted interest rate factor $V(\delta_0, s)$ follows a lognormal distribution. Now define a new function to express the sum of the forces of interest:

Definition 3.1 The sum of the forces of interest is denoted by

$$I(\delta_0, s) = \begin{cases} \sum_{i=0}^{[s]-1} ln(1+\delta_i) + \{s\} ln\left(1+\delta_{[s]}\right), & s \geq 1 \\ s \cdot ln(1+\delta_0), & 0 \leq s < 1 \end{cases},$$

where $[s]$ stands for the integer part of s and $\{s\}$ stands for the decimal part of s.

Using the terms in Definition 4.1, we can further obtain the expressions of the expected value and the variance of $V(\delta_0, s)$ as well as the covariance terms of $V(\delta_0, s)$ and $V(\delta_0, u)$ (refer to Propositions 3.3 - 3.5):

Proposition 3.3 The expected value of the discounted interest rate factor over s year(s) is

$$E[V(\delta_0, s)] = E\left[e^{-I(\delta_0, s)}\right] = \begin{cases} exp\left[-sln(1+\delta_0) + \frac{\sigma_a^2}{2} c(s)\right], & s \geq 1 \\ \left(\frac{1}{1+\delta_0}\right)^s, & 0 \leq s < 1 \end{cases}, \tag{3.18}$$

where

$$\begin{cases} c(s) = \{s\}^2 + \frac{[s]-1}{(1-\phi)^2} + \frac{2b(s)\left(\phi-\phi^{[s]}\right)}{(1-\phi)^2} + [b(s)]^2 \frac{\phi^2-\phi^{2[s]}}{1-\phi^2}, \\ b(s) = \{s\} - \frac{1}{1-\phi}, \end{cases} \quad s \geq 1.$$

Proof. Obviously, (3.18) holds when $0 \leq s < 1$. For $s \geq 1$, we express the expected value of the discounted interest rate factor, $V(\delta_0, s)$, in terms of the sum of the force of interest, $I(\delta_0, s)$:

$$\begin{aligned} E[V(\delta_0, s)] &= E\left[e^{-I(\delta_0, s)}\right] \\ &= exp\left\{-E\left[I(\delta_0, s)\right] + \frac{1}{2} Var\left[I(\delta_0, s)\right]\right\} \\ &= exp\left\{-sln(1+\delta_0) + \frac{1}{2} Var\left[\sum_{i=1}^{[s]-1} X_i + \{s\} X_{[s]}\right]\right\}, \end{aligned} \tag{3.19}$$

where

$$\begin{cases} X_i = ln(1+\delta_i) - ln(1+\delta_0) \\ a_k = X_k - \phi X_{k-1} \end{cases} \tag{3.20}$$

for $i \in \mathbb{N}$ and $k \in \mathbb{N}^+$.

$$\begin{aligned}
&Var\left[\sum_{i=1}^{[s]-1} X_i + \{s\}X_{[s]}\right] \\
&= Var\left[\sum_{i=1}^{[s]-1}\sum_{j=0}^{i-1} a_{i-j}\phi^j + \{s\}\sum_{j=0}^{[s]-1} a_{[s]-j}\phi^j\right] \\
&= Var\left[\sum_{j=1}^{[s]-1} a_j a(s,j) + \{s\}a_{[s]}\right] \\
&= \sigma_a^2\left\{\sum_{j=1}^{[s]-1}[a(s,j)]^2 + \{s\}^2\right\} \\
&= \sigma_a^2\left\{\{s\}^2 + \frac{[s]-1}{(1-\phi)^2} + \frac{2b(s)\left(\phi-\phi^{[s]}\right)}{(1-\phi)^2} + [b(s)]^2\frac{\phi^2-\phi^{2[s]}}{1-\phi^2}\right\}, \qquad (3.21)
\end{aligned}$$

where $a(s,j) = b(s)\phi^{[s]-j} + \frac{1}{1-\phi}$ and $b(s) = \{s\} - \frac{1}{1-\phi}$ for $s \geq 1$. Therefore, (3.19) can be rewritten as

$$E[V(\delta_0, s)] = exp\left\{-sln(1+\delta_0) + \frac{\sigma_a^2}{2}c(s)\right\}, \qquad s \geq 1,$$

where

$$c(s) = \{s\}^2 + \frac{[s]-1}{(1-\phi)^2} + \frac{2b(s)\left(\phi-\phi^{[s]}\right)}{(1-\phi)^2} + [b(s)]^2\frac{\phi^2-\phi^{2[s]}}{1-\phi^2}.$$

□

Proposition 3.4 The variance of the discounted interest rate factor over s years is

$$Var\left[V(\delta_0,s)\right] = \begin{cases} exp\left\{-2sln(1+\delta_0) + \sigma_a^2 c(s)\right\}\left(e^{\sigma_a^2 c(s)} - 1\right), & s \geq 1 \\ 0, & 0 \leq s < 1 \end{cases}, \qquad (3.22)$$

where

$$c(s) = \{s\}^2 + \frac{[s]-1}{(1-\phi)^2} + \frac{2b(s)\left(\phi-\phi^{[s]}\right)}{(1-\phi)^2} + [b(s)]^2\frac{\phi^2-\phi^{2[s]}}{1-\phi^2}, \qquad s \geq 1.$$

Proof. Since we have assumed that the annual interest rate is known for the first year, it is very obvious that (3.22) holds for $0 \leq s < 1$.

For $s \geq 1$, we express the second moment of the discounted interest rate factor, $E\left[V^2(\delta_0, s)\right]$, in terms of the sum of the forces of interest, $I(\delta_0, s)$:

$$\begin{aligned} E\left[V^2(\delta_0, s)\right] &= E\left(e^{-2I(\delta_0,s)}\right) \\ &= exp\left\{-2E\left[I(\delta_0, s)\right] + \frac{1}{2}Var\left[2I(\delta_0, s)\right]\right\} \\ &= exp\left\{-2sln(1+\delta_0) + 2Var\left[I(\delta_0, s)\right]\right\} . \end{aligned}$$

Furthermore, by using the conclusion in Proposition 3.3, we have the expression of the variance of the discounted interest rate factor as follows:

$$\begin{aligned} Var\left[V(\delta_0, s)\right] &= E\left[V^2(\delta_0, s)\right] - \left\{E\left[V(\delta_0, s)\right]\right\}^2 \\ &= E\left[e^{-2I(\delta_0,s)}\right] - E\left[e^{-2I(\delta_0,s)}\right]^2 \\ &= exp\left\{-2sln(1+\delta_0) + 2\sigma_a^2 c(s)\right\} - exp\left\{-2sln(1+\delta_0) + \sigma_a^2 c(s)\right\} \\ &= exp\left\{-2sln(1+\delta_0) + \sigma_a^2 c(s)\right\}\left(e^{\sigma_a^2 c(s)} - 1\right) , \end{aligned}$$

where

$$\left\{ \begin{array}{l} c(s) = \{s\}^2 + \frac{[s]-1}{(1-\phi)^2} + \frac{2b(s)\left(\phi - \phi^{[s]}\right)}{(1-\phi)^2} + [b(s)]^2\, \frac{\phi^2 - \phi^{2[s]}}{1-\phi^2} , \\ b(s) = \{s\} - \frac{1}{1-\phi} , \end{array} \right. \quad s \geq 1 .$$

□

Proposition 3.4 is just a special case of the covariance function. To get the covariance terms for the discounted interest rate factors at different time moments, we need to find the value of the cross expectation terms, $E[V(\delta_0, s)V(\delta_0, u)]$ for $s, u \geq 0$ and $s \neq u$.

By symmetry, we have $E[V(\delta_0, s)V(\delta_0, u)] = E[V(\delta_0, u)V(\delta_0, s)]$. Thus, we only need to discuss the situation where $s < u$. Set $s = t_0 k$ and $u = t_0 i$ for $0 < t_0 \leq 1$ and $k, i \in \mathbb{N}$, where t_0 implies the frequency of the status check. For our model in this thesis, $t_0 = 0.5$. The derivations for the special case of $t_0 = 1$ (i.e. s and u are integer-valued) was given in Chen (2010).

Proposition 3.5 For $0 \leq s < u$, the auto-covariance function of the discounted interest rate process is

$$\begin{aligned} &Cov\left[V(\delta_0, s), V(\delta_0, u)\right] \\ &= \left\{ \begin{array}{ll} exp\left\{-(s+u)ln(1+\delta_0) + \frac{\sigma_a^2}{2}\left[c(s) + c(u)\right]\right\}\left(e^{\sigma_a^2 d(s,u)} - 1\right) , & 1 \leq s < u \\ 0 , & otherwise \end{array} \right. , \end{aligned} \tag{3.23}$$

where

$$\begin{cases} d(s,u) = \sum_{j=1}^{[s]-1} a(s,j)a(u,j) + \{s\}a\left(u,[s]\right) \,, \\ c(s) = \{s\}^2 + \frac{[s]-1}{(1-\phi)^2} + \frac{2b(s)\left(\phi-\phi^{[s]}\right)}{(1-\phi)^2} + [b(s)]^2\,\frac{\phi^2-\phi^{2[s]}}{1-\phi^2}\,, \\ b(s) = \{s\} - \frac{1}{1-\phi}\,, \\ a(s,u) = b(s)\phi^{[s]-u} + \frac{1}{1-\phi}\,. \end{cases} \tag{3.24}$$

Proof. Firstly, for $0 \leq s < 1\,, s < u$, we have

$$\begin{aligned} Cov\left[V(\delta_0,s),V(\delta_0,u)\right] =& E\left[V(\delta_0,s)V(\delta_0,u)\right] - E\left[V(\delta_0,s)\right]E\left[V(\delta_0,u)\right] \\ =& V(\delta_0,s)E\left[V(\delta_0,u)\right] - E\left[V(\delta_0,s)\right]E\left[V(\delta_0,u)\right] \\ =& \left(\frac{1}{1+\delta_0}\right)^s E\left[V(\delta_0,u)\right] - \left(\frac{1}{1+\delta_0}\right)^s E\left[V(\delta_0,u)\right] \\ =& 0\,. \end{aligned}$$

For $1 \leq s < u$,

$$\begin{aligned} E[V(\delta_0,s)V(\delta_0,u)] =& E\left\{e^{-[(I(\delta_0,s)I(\delta_0,u)]}\right\} \\ =& exp\left\{-E\left[I(\delta_0,s)+I(\delta_0,u)\right] + \frac{1}{2}Var\left[I(\delta_0,s)+I(\delta_0,u)\right]\right\} \\ =& exp\left\{-E\left[I(\delta_0,s)\right] - E\left[I(\delta_0,u)\right] + \frac{1}{2}Var\left[I(\delta_0,s)\right]\right. \\ & \left. + \frac{1}{2}Var\left[I(\delta_0,u)\right] + Cov\left[I(\delta_0,s),I(\delta_0,u)\right]\right\} \\ =& exp\left\{ -(s+u)ln(1+\delta_0) + \frac{1}{2}Var\left[I(\delta_0,s)\right] + \frac{1}{2}Var\left[I(\delta_0,u)\right]\right. \\ & \left. + Cov\left[I(\delta_0,s),I(\delta_0,u)\right]\right\}, \end{aligned} \tag{3.25}$$

where $Var[I(\delta_0, s)]$ is given in (3.21) and the covariance term can be obtained by

$$\begin{aligned}
&Cov\left[I(\delta_0, s), I(\delta_0, u)\right]\\
=&Cov\left[\left(\sum_{i=1}^{[s]-1} X_i + \{s\}X_{[s]}\right), \left(\sum_{i=1}^{[u]-1} X_i + \{u\}X_{[u]}\right)\right]\\
=&Cov\left\{\left[\sum_{j=1}^{[s]-1} a_j a(s,j) + \{s\}a_{[s]}\right], \left[\sum_{j=1}^{[u]-1} a_j a(u,j) + \{u\}a_{[u]}\right]\right\}\\
=&Cov\left\{\left[\sum_{j=1}^{[s]-1} a_j a(s,j) + \{s\}a_{[s]}\right], \left[\sum_{j=1}^{[s]} a_j a(u,j)\right]\right\}\\
=&\sigma_a^2\left[\sum_{j=1}^{[s]-1} a(s,j)a(u,j) + \{s\}a(u,[s])\right]\\
=&\sigma_a^2 d(s,u)\,, \qquad 0 \leq s < u\,,
\end{aligned} \tag{3.26}$$

where

$$d(s,u) = \sum_{j=1}^{[s]-1} a(s,j)a(u,j) + \{s\}a\left(u,[s]\right)$$

for $1 \leq s < u$. Therefore, the covariance function can be rewritten as

$$\begin{aligned}
&Cov\left[V(\delta_0, s), V(\delta_0, u)\right]\\
=&E\left[V(\delta_0, s), V(\delta_0, u)\right] - E\left[V(\delta_0, s)\right]E\left[V(\delta_0, u)\right]\\
=&exp\left\{-(s+u)ln(1+\delta_0) + \frac{\sigma_a^2}{2}\left[c(s) + c(u) + 2d(s,u)\right]\right\}\\
&- exp\left\{-(s+u)ln(1+\delta_0) + \frac{\sigma_a^2}{2}\left[c(s) + c(u)\right]\right\}\\
=&exp\left\{-(s+u)ln(1+\delta_0) + \frac{\sigma_a^2}{2}\left[c(s) + c(u)\right]\right\}\left(e^{\sigma_a^2 d(s,u)} - 1\right)\,, \qquad 1 \leq s < u\,.
\end{aligned}$$

For those intermediate terms used for calculations, please refer to (3.24). □

By symmetry, we can obtain a more general formula for the covariance function for $s \neq u$ and $s\,,u > 0$ as follows:

$$\begin{aligned}
&Cov\left[V(\delta_0, s), V(\delta_0, u)\right]\\
&= \sigma_a^2\left[\sum_{j=1}^{min([s],[u])-1} a(s,j)a(u,j)\right] + \{min(s,u)\}a\left(max(s,u), min\left([s],[u]\right)\right)
\end{aligned} \tag{3.27}$$

for $s\,,u > 0$ and $s \neq u$.

3.2 Mortality Assumptions

In this section, we shall introduce the mortality assumptions and tables used to calculate the transition intensities among the four states for the long-term disability insurance model introduced in Section 2.3.

The main purpose of this insurance product is to provide funds for employees in case of disability or death, so we apply the service table and the mortality table provided in Bowers et al.∼ (1997).

Definition 3.2 Define the **transition intensity** (or the force of transition), $\mu^{ij}(x)$ as follows:

$$\mu^{ij}(x) = \lim_{h \to 0^+} \frac{{}_h p^{ij}(x)}{h}\,, \qquad h > 0\,, x \in \mathbb{N}^+\,, i\,, j \in \Xi\,, i \neq j\,,$$

where ${}_h p^{ij}(x)$ is the transition probability of the continuous-time Markov chain process, $\{M_t(x); t \geq 0\}$ defined in Definition 2.1 and

$$\mu^{ij}(x+t) = \mu^{ij}(x)\,,$$

for any $t \in [0, 1)$ and any $x \in \mathbb{N}^+$.

Furthermore, we make the piecewise constant force assumption for the transition process within each age interval $[x, x + 1)$ for convenience purpose. In other words, the transition intensity, denoted by $\mu^{ij}(x + t)$, is a constant for $t \in [0, 1)$. The relationships between the transition intensity and the transition probability in calculating the transition matrix shall be presented in Chapter 4.

In addition to the piecewise constant forces of transition assumption for each age interval, we make some assumptions to describe the relationships between the constant forces of transition, $\mu^{ij}(x + t)\,, i\,, j \in \Xi$. According to a disability survey in Japan presented in Rajnes (2010) and the OPCS data studied in Martin et al.∼ (1988) used by Haberman et al.∼ (1997), we make several further assumptions as follows:

- The chance of recovery for disability is 10% of that of becoming temporarily disabled: $\mu^{10}(x) = 0.1\mu^{01}(x)$;

- the mortality rate for healthy, temporarily disabled and permanently disabled people are the same: $\mu^{03}(x) = \mu^{13}(x) = \mu^{23}(x)$;
- the chance of turning permanently disabled for an insured who is temporarily disabled is at the same level as that for a healthy insured: $\mu^{12}(x) = \mu^{02}(x)$;
- among all the disabled people, the chance of being permanently disabled is 1.5 times as that of being temporarily disabled: $\mu^{02}(x) = 1.5\mu^{01}(x)$.

3.3 Other Assumptions

In later chapters, we shall study the future lifetime and the status of an insured and the present value of the future benefits generated from an insurance portfolio. We define $Z(x, n, FV)$ the variable of the present value of the future benefit payments (PVFBP) to the insured who is insured by the n-year term disability insurance plan at age x with face value FV.

For this insurance portfolio, we further assume that the transition processes are all i.i.d among all the insureds and the interest rate process is independent of the transition process for the insureds.

In Chapters 4 and 5, we shall apply all the assumptions stated in this chapter to calculate the transition intensities and the moments of the PVFBP under the disability insurance policy, and analyze the risk for a single policy and for a portfolio.

Chapter 4

Valuation of Single Policies

In this chapter, we firstly introduce a long-term disability insurance product and then provide the methodology for calculating the transition probabilities used for valuations. The Kolmogorov's forward equation is also presented as an alternative method for calculation purpose. In Sections 4.2 and 4.3, the recursive method and the cash flow method are given for moment calculation of the Present Value of the Future Benefit Payment (PVFBP).

4.1 The Markov Chain Model for Transition Probability Calculation

The long-term disability insurance model consists of four states as was described in Section 3.4: healthy, temporarily disabled, permanently disabled and deceased. The policy for the n-year term disability insurance product is given as follows:

1. the status check is done semiannually;
2. semiannual benefit payment b_1 (we set b_1 to be the face value (FV) of this policy) is paid while temporarily disabled and before the maturity date of the policy;
3. semiannual benefit payment $b_2 = 2FV$ is paid while permanently disabled and before the maturity date of the policy;
4. a lump sum payment of $b_3 = 30FV$ is paid upon the earliest moment between death and the maturity date of the policy.

In the next section, we shall introduce assumptions and propositions for calculating the transition process (see Section 8.3 in Dickson et al.∼ (2009)).

4.1.1 The Assumptions and Propositions

Assumption 1. The transition process for the long-term disability insurance model is a continuous time non-homogeneous Markov chain process. In other words, the future transition probabilities only depend on the previous state and age of the policyholder, but not depend on the duration that the policyholder has stayed in a certain state.

Assumption 2. The transition probability, ${}_hp^{ij}(x)$, is a differentiable function of h for $i, j \in \Xi\,, h > 0$ and $x \in \mathbb{N}$. This property will guarantee the existence of the force of transition, $\mu^{ij}(x)$, by its definition.

Proposition 4.1. For a small positive value of h,

$$ {}_hp^{ij}(x) = h\mu^{ij}(x) + o(h)\,, \qquad i \neq j\,, i, j \in \Xi\,. \tag{4.1} $$

Proof. It is easy to derive this proposition by rewriting the definition of the force of transition, $\mu^{ij}(x)$, in Definition 3.2. □

Furthermore, we have

$$ {}_hp^{ij}(x) \approx h\mu^{ij}(x)\,, \qquad i \neq j\,, i, j \in \Xi\,, \tag{4.2} $$

for small positive value of h.

Lemma 4.1. For a small positive value of h, we have the following transition probability that the insured never leaves state i between age x and $x + h$, denoted as ${}_hp^{\bar{i}\bar{i}}(x)$:

$$ {}_hp^{\bar{i}\bar{i}}(x) = 1 - h \sum_{j=0, j\neq i}^{3} \mu^{ij}(x) + o(h)\,, \qquad i\,, j \in \Xi\,, h > 0\,, x \in \mathbb{N}^+\,. \tag{4.3} $$

Proof. From the conclusion of Proposition 4.1, we have

$$ h \sum_{j=0, j\neq i}^{3} \mu^{ij}(x) + o(h) = \sum_{j=0, j\neq i}^{3} {}_hp^{ij}(x)\,, \qquad i\,, j \in \Xi\,, h > 0\,, x \in \mathbb{N}^+\,. \tag{4.4} $$

For a small positive value of h, we have

$$ {}_hp^{\bar{i}\bar{i}}(x) + o(h) = {}_hp^{ii}(x) = 1 - \sum_{j=0,j\neq i}^{3} {}_hp^{ij}(x)\,, $$

where ${}_hp^{ii}(x)$ is the probability that given the information that the insured is in state i at age x, the probability of the same insured is in state i at age $x+h$. Furthermore, by (4.4), we have

$$ {}_hp^{\bar{i}\bar{i}}(x) = 1 - h \sum_{j=0,j\neq i}^{3} \mu^{ij}(x) + o(h)\,, \qquad i\,,j \in \Xi\,, h > 0\,, x \in \mathbb{N}^+\,. $$

□

Proposition 4.2. For a small positive value of h, the transition probability ${}_hp^{\bar{i}\bar{i}}(x)$ can be expressed as

$$ {}_hp^{\bar{i}\bar{i}}(x) = exp\left\{ -\int_0^h \sum_{j=0,j\neq i}^{3} \mu^{ij}(x+s)ds \right\}\,, \qquad i\,,j \in \Xi\,, h > 0\,, x \in \mathbb{N}^+\,. \tag{4.5} $$

Proof. Firstly, we have

$$ {}_hp^{\bar{i}\bar{i}}(x+t) = \frac{{}_{t+h}p^{\bar{i}\bar{i}}(x)}{{}_tp^{\bar{i}\bar{i}}(x)} $$

by definition. By applying (4.3), we can write

$$ {}_hp^{\bar{i}\bar{i}}(x+t) = 1 - h \sum_{j=0,j\neq i}^{3} \mu^{ij}(x+t) + o(h)\,. \tag{4.6} $$

Multiplying both sides of (4.6) by $1/h$ and rearranging, we get

$$ \frac{{}_{t+h}p^{\bar{i}\bar{i}}(x) - {}_tp^{\bar{i}\bar{i}}(x)}{h \cdot {}_tp^{\bar{i}\bar{i}}(x)} = - \sum_{j=0,j\neq i}^{3} \mu^{ij}(x+t) + \frac{o(h)}{h}\,. $$

Letting $h \to 0$, we have

$$ \frac{d\left\{ln\left[{}_tp^{\bar{i}\bar{i}}(x)\right]\right\}}{dt} = - \sum_{j=0,j\neq i}^{3} \mu^{ij}(x+t)\,. $$

By integration over the interval $[0, h]$ for $h > 0$, we have

$$ln\left[{}_h p^{\bar{i}\bar{i}}(x)\right] - ln\left[{}_0 p^{\bar{i}\bar{i}}(x)\right] = -\int_0^h \sum_{j=0,j\neq i}^{3} \mu^{ij}(x+s)ds\,.$$

Finally, taking the exponential on both sides of this equation, we can easily obtain the result shown in (4.5). Furthermore, under the constant force of transition assumption, we have for $0 \le h \le 0.5$, the conclusion of Proposition 4.2 can be rewritten as

$${}_h p^{\bar{i}\bar{i}}(x) = exp\left\{-h \sum_{j=0,j\neq i}^{3} \mu^{ij}(x)\right\}, \qquad i\,,j \in \Xi\,, h > 0\,, x \in \mathbb{N}^+\,.$$

□

Now by (4.2) and (4.3) for a small positive value of h, the further issue to be taken into account is how small the interval should be to guarantee the accuracy of the transition probabilities. We can check the reasonableness of the frequency of status check, t_0, by adding each row of the transition matrix and see if the row sum is very close to 1. In the previous chapters, we have set $t_0 = 0.5$, which implies a semiannual status check frequency.

When $t_0 = 0.5$, the one-step transition probability can be expressed in terms of the forces of transition (or transition intensity):

$$\begin{aligned} {}_{0.5}p^{ij}(x) &= \begin{cases} e^{-\int_0^{0.5} \sum_{k\neq i} \mu^{ik}(x+t)dt}, & i = j \\ h\mu^{ij}(x)\,, & i \neq j \end{cases} \\ &= \begin{cases} e^{-0.5 \sum_{k\neq i} \mu^{ik}(x)}, & i = j \\ 0.5\mu^{ij}(x)\,, & i \neq j \end{cases}, \qquad i\,,j \in \Xi\,, x \in \mathbb{N}^+\,. \end{aligned} \tag{4.7}$$

By following the testing method above, we conclude that the value of $t_0 = 0.5$ is s-mall enough to guarantee the accuracy of the approximation of the transition probabilities. Now, the one-period transition probability for the continuous time Markov chain process $\{M_t(x)\,; t \ge 0\}$ can be obtained in terms of the forces of transition:

$$\begin{aligned} Q^{ij}_{(x,1,0.5)} &= {}_{0.5}p^{ij}(x) \\ &= \begin{cases} e^{-0.5 \sum_{k\neq i} \mu^{ik}(x)}, & i = j \\ 0.5\mu^{ij}(x)\,, & i \neq j \end{cases}, \qquad i\,,j \in \Xi\,, x \in \mathbb{N}^+\,. \end{aligned} \tag{4.8}$$

By applying the general formula for a one-period transition probability demonstrated in (4.7), we give a numerical example of the one-period transition matrix for a 40-year-old insured below:

$$Q_{(40,1,0.5)} = \begin{bmatrix} 0.998240 & 0.00028 & 0.000423 & 0.001057 \\ 0.000028 & 0.998494 & 0.000423 & 0.001057 \\ 0 & 0 & 0.998944 & 0.001057 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

of which the row sums are 1.000002 , 1.000001 , 1.000001 and 1 respectively. Obviously, those values are fairly close to 1. Furthermore, the general formula of the k-step transition matrix under the long-term disability insurance model for an x-year-old insured can be obtained approximately as follows:

$$\begin{aligned} & Q_{(x,k,0.5)} \\ &= \prod_{i=0}^{k-1} Q_{(x+0.5i,1,0.5)} \\ &= \prod_{i=0}^{k-1} \left\{ \begin{bmatrix} e^{-0.5\sum_{j=1}^{3}\mu^{0j}(x+0.5i)} & 1-e^{-0.5\mu^{10}(x+0.5i)} & 0 & 0 \\ 1-e^{-0.5\mu^{01}(x+0.5i)} & e^{-0.5\sum_{j=0,j\neq 1}^{3}\mu^{1j}(x+0.5i)} & 0 & 0 \\ 1-e^{-0.5\mu^{02}(x+0.5i)} & 1-e^{-0.5\mu^{12}(x+0.5i)} & e^{-0.5\mu^{23}(x+0.5i)} & 0 \\ 1-e^{-0.5\mu^{03}(x+0.5i)} & 1-e^{-0.5\mu^{13}(x+0.5i)} & 1-e^{-0.5\mu^{23}(x+0.5i)} & 1 \end{bmatrix}^T \right\}, \end{aligned} \tag{4.9}$$

where $k\,, x \in \mathbb{N}^+\,, j \in \Xi$. Alternatively, it can be calculated by the Kolmogorov's forward equation introduced in the next section.

4.1.2 The Kolmogorov's Forward Equation

In this section, we shall extend the results of Proposition 4.1 and 4.2 to a more general formula, which is known as **The Kolmogorov's Forward Equation** (see Chapter 8 of Dickson et al.$\sim$ (2009)):

Proposition 4.3. (The Kolmogorov's Forward Equation) For $t \geq 0$, $x \in \mathbb{N}$, and a small positive value of h, we have

$$ {}_{t+h}p^{ij}(x) = {}_{t}p^{ij}(x) - h \sum_{k=0,k\neq j}^{3} \Big[{}_{t}p^{ij}(x)\mu^{jk}(x+t) - {}_{t}p^{ik}(x)\mu^{kj}(x+t) \Big] + o(h)\,, \qquad (4.10) $$

and thus the following equation holds for the transition probabilities (see Dickson et al.∼ (2009)):

$$ \frac{d\left[{}_{t}p^{ij}(x)\right]}{dt} = \sum_{k=0,k\neq j}^{3} \Big[{}_{t}p^{ik}(x)\mu^{kj}(x+t) - {}_{t}p^{ij}(x)\mu^{jk}(x+t) \Big]\,, t \geq 0\,, x \in \mathbb{N}^{+}\,. \qquad (4.11) $$

Proof. By definition, we have

$$ {}_{t+h}p^{ij}(x) = {}_{t}p^{ij}(x)\,{}_{h}p^{jj}(x+t) + \sum_{k=0,k\neq j}^{3} {}_{t}p^{ik}(x)\,{}_{h}p^{kj}(x+t)\,, t\,, h \geq 0\,, x \in \mathbb{N}\,. $$

Then by the conclusions in Propositions 4.1 and 4.2, we can rewrite the formula as

$$ \begin{aligned} & {}_{t+h}p^{ij}(x) \\ &= {}_{t}p^{ij}(x)\left[1 - h\sum_{k=0,k\neq j}^{3}\mu^{jk}(x+t) + o(h)\right] + h\sum_{k=0,k\neq j}^{3} {}_{t}p^{ik}(x)\mu^{kj}(x+t) + o(h) \\ &= {}_{t}p^{ij}(x) + h\sum_{k=0,k\neq j}^{3}\Big[{}_{t}p^{ik}(x)\mu^{kj}(x+t) - {}_{t}p^{ij}(x)\mu^{jk}(x+t)\Big] + o(h)\,, \end{aligned} \qquad (4.12) $$

for $t\,, h \geq 0\,, x \in \mathbb{N}^{+}$. □

Basically, (4.11) sets a system of Kolmogorov's Forward Equations for the Markov chain process. Letting $t = t_0(l-1)$ and $h = t_0$ in (4.12), the transition probabilities after l periods can be expressed as:

$$ \begin{aligned} & Q^{ij}_{(x,l,t_0)} \\ &= {}_{t_0 l}p^{ij}(x) \\ &\approx {}_{t_0(l-1)}p^{ij}(x) + t_0\sum_{k=0,k\neq j}^{3}\Big[{}_{t_0(l-1)}p^{ik}(x)\mu^{kj}\left(x+t_0(l-1)\right) - {}_{t_0(l-1)}p^{ij}(x)\mu^{jk}\left(x+t_0(l-1)\right)\Big] \end{aligned} $$

for $t_0 > 0\,, x\,, l \in \mathbb{N}^{+}$ and $i\,, j\,, k \in \Xi$.

4.2 Recursive Formula for Moment and Calculations

In this section, we derive a recursive formula to calculate the moments of the present value of future benefit payments (PVFBP) by applying the results in Proposition 4.3.

Define $Z(x, n, FV)$ the variable of the future benefit payments to an n-year term disability insurance for an insured who joins the plan at age x with temporary disability benefit payment FV. Following the policy rule stated at the beginning of Section 4.1, we can calculate the moment of the PVFBP by considering two different situations:

Situation 1: No recovery from the disability (if applies) during the term of the contract

For this situation, we define t_1 the time of becoming temporarily disabled, t_2 the time of becoming permanently disabled and t_3 the time of death. Figures 4.1 - 4.4 illustrates four possible scenarios under this situation:

1. Healthy→Temporarily Disabled→Permanently Disabled→Deceased (i.e. $t_1 < t_2 < t_3$)
2. Healthy→Temporarily Disabled→Deceased (i.e. $t_1 < t_2 = t_3$)
3. Healthy→Permanently Disabled→Deceased (i.e. $t_1 = t_2 < t_3$)
4. Healthy→Deceased (i.e. $t_1 = t_2 = t_3$)

Figure 4.1: The Long-Term Disability Insurance Model: Transition Scenario 1

Situation 2: At least one recovery from disability during the term of contract

Under this situation (see Figure 4.5), we define t'_1 the first time becoming temporarily

Figure 4.2: The Long-Term Disability Insurance Model: Transition Scenario 2

Figure 4.3: The Long-Term Disability Insurance Model: Transition Scenario 3

disabled and t'_2 the first time recovering from disabled after age x. For both situations above, $t_1, t_2, t_3, t'_1, t'_2$ are all multipliers of the checking frequency t_0 ($t_0 = 0.5$ in this thesis).

Moreover, for both situations above, we define T_1, T_2, T_3, T'_1 and T'_2 the relevant random variables of those turning time points, the possible values of which are denoted as t_1, t_2, t_3, t'_1 and t'_2. Then considering the future benefit payments generated from the two situations, we can obtain a recursive expression of the first moment of $E\{[Z(x, n, FV)]\}$ by summing up the expectations under the two situations, denoted by $E_1\{[Z(x, n, FV)]\}$ and $E_2\{[Z(x, n, FV)]\}$, respectively.

Considering the four scenarios shown in Situation 1, if FV stands for the temporary disability benefit payment amount, then the corresponding part of the future benefit payments

Figure 4.4: The Long-Term Disability Insurance Model: Transition Scenario 4

Figure 4.5: The Long-Term Disability Insurance Model: Transition Scenario 5

for this insured can be obtained by:

$$\begin{aligned}
&E_1\left\{[Z(x,n,FV)]^m\right\}\\
&=E_V\Bigg\{\sum_{t_1=0.5}^{n}\sum_{t_2=t_1}^{n}\sum_{t_3=t_2}^{n+0.5}Pr(T_1=t_1,T_2=t_2,T_3=t_3)\\
&\quad\times\Bigg[I_{\{t_2>t_1\}}\sum_{k=t_1}^{t_2-0.5}[FV\cdot V(\delta_0,k)]^m+I_{\{t_3>t_2\}}\sum_{k=t_2}^{t_3-0.5}[2FV\cdot V(\delta_0,k)]^m\\
&\quad+I_{\{t_3\le n\}}[30FV\cdot V(\delta_0,t_3)]^m\Bigg]\Bigg\}
\end{aligned}\tag{4.13}$$

where E_V denotes the expectation with respect to the interest rate process and

$$\begin{aligned}
&Pr(T_1 = t_1, T_2 = t_2, T_3 = t_3) \\
=&I_{\{t_1<t_2\}} \times I_{\{t_2<t_3\}} \times q'_{01}(x, t_1) \times q'_{12}(x + t_1, t_2 - t_1) \times q'_{23}(x + t_2, t_3 - t_2) \\
&+ I_{\{t_1<t_2\}} \times I_{\{t_2=t_3\}} \times q'_{01}(x, t_1) \times q'_{13}(x + t_2, t_3 - t_1) \\
&+ I_{\{t_1=t_2\}} \times I_{\{t_2<t_3\}} \times q'_{02}(x, t_2) \times q'_{23}(x + t_2, t_3 - t_2) + I_{\{t_1=t_2\}} \times I_{\{t_2=t_3\}} \times q'_{03}(x, t_3) \,,
\end{aligned} \tag{4.14}$$

and $q'_{ij}(x,t)$ stands for the probability that an insured starting from state i at age x (first insured age), stays in state i for $(t-0.5)$ years and then transfers to state j at age $x+t$. In other words,

$$q'_{ij}(x,t) = \begin{cases} Q^{ii}_{(x,2t-1,0.5)} \times Q^{ij}_{(x+t-0.5,1,0.5)} \,, & t > 0.5 \\ Q^{ij}_{(x,1,0.5)} \,, & t = 0.5 \end{cases} . \tag{4.15}$$

Note that the four terms in (4.14) correspond to the scenarios $(a), (b), (c), (d)$ shown in Figures 4.1 - 4.4 respectively. Under **Situation 2** above, an x-year-old insured becomes temporarily disabled at a certain time point in the age interval $[x, x+t)$ and then recovers from the disability at age $x+t'$, we assume that the transition processes before and after age $x+t$ are independent conditioning on interest rates. In other words, if we define Y_1 and Y_2 to be the random variables of the present value of the future benefit payments before and after the first time recovering from temporarily disabled after age x. Then by the assumption stated above, we have

$$\begin{aligned}
E\left[Y_1^i Y_2^j\right] &= E_V\left\{E\left[Y_1^i Y_2^j | \{\delta_k; k \in \mathbb{N}\}\right]\right\} \\
&= E_V\left\{E\left[Y_1^i | \{\delta_k; k \in \mathbb{N}\}\right]] E\left[Y_2^j | \{\delta_k; k \in \mathbb{N}\}\right]\right\} \,, \qquad i\,, j \in \mathbb{N}^+ \,,
\end{aligned} \tag{4.16}$$

where $\{\delta_k; k \in \mathbb{N}\}$ refers to the interest rate process from time 0.

Then a recursive formula to estimate the second part of the first two moments of the future benefit payment with chances of recovery can be expressed in terms of the moments of an insurance with shorter terms:

$$\begin{aligned}
E_2\left[Z(x,n,FV)\right] =& E_V\left[E\left(Y_1+Y_2|\{\delta_k;k\in\mathbb{N}\}\right)\right]\\
=& E_V\left\{\sum_{t_1'=0.5}^{n-0.5}\sum_{t_2'=t_1'+0.5}^{n} q_{01}'(x,t_1')q_{10}'(x+t_1',t_2'-t_1')\right.\\
&\left.\times\left\{\sum_{k=t_1'}^{t_2'-0.5} FV\cdot V\left(\delta_0,k\right)+V(\delta_0,t_2')E\left[Z(x+t_2',n-t_2',FV)\right]\right\}\right\}\\
=& \sum_{t_1'=0.5}^{n-0.5}\sum_{t_2'=t_1'+0.5}^{n} q_{01}'(x,t_1')q_{10}'(x+t_1',t_2'-t_1')\\
&\times\left\{\sum_{k=t_1'}^{t_2'-0.5} FV\cdot E\left[V\left(\delta_0,k\right)\right]+E\left[V(\delta_0,t_2')\right]E\left[Z(x+t_2',n-t_2',FV)\right]\right\}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&E_2\left[Z^2(x,n,FV)\right]\\
=&E_V\left[Y_1^2+Y_2^2+2Y_1Y_2|\{\delta_k;k\in\mathbb{N}\}\right]\\
=&E_V\left\{\sum_{t_1'=0.5}^{n-0.5}\sum_{t_2'=t_1'+0.5}^{n} q_{01}'(x,t_1')q_{10}'(x+t_1',t_2'-t_1')\right.\\
&\times\left\{\left[FV\cdot\sum_{k=t_1'}^{t_2'-0.5}V\left(\delta_0,k\right)\right]^2+\left[V(\delta_0,t_2')\right]^2E\left[Z^2(x+t_2',n-t_2',FV)\right]\right.\\
&\left.\left.+2V\left(\delta_0,t_2'\right)E\left[Z(x+t_2',n-t_2',FV)\right]\left[FV\cdot\sum_{k=t_1'}^{t_2'-0.5}V\left(\delta_0,k\right)\right]\right\}\right\}
\end{aligned} \tag{4.18}$$

for $t_1', t_2' \geq 0, n > 0.5$, and $x \in \mathbb{N}^+$. In addition, we have

$$E\left[Z^m(x,n,FV)\right] = E_1\left[Z^m(x,n,FV)\right], \qquad m=1,2, n\leq 0.5, FV>0, x\in\mathbb{N}^+, \tag{4.19}$$

since we have only one observation for a half-year term policy and observations of recovery is impossible. Now with the starting values, $Z(x, 0.5, FV)$ known, one can obtain $Z(x, 0.5n, FV)$ recursively ($n > 1$) by applying (4.17) and (4.18).

4.3 Cash Flow Method for Moment Calculations

In addition to the recursive procedure introduced in the previous section, there is a more direct and efficient way to obtain the moments of the PVFBP for an insured and further to analyze the insurance risk and investment risk of an insurance portfolio, which is known as the **Cash Flow Method**. In this section, we simply demonstrate the framework of this method for the moment calculation of the long-term disability insurance product.

The main idea of the cash flow method in the context of traditional life insurance moment calculation applications (e.g. term life insurance, endowment) is to first calculate the possibility of a survival/death payment at any future payment time t, and then get the expectation of the cash flow on payment at time t. Finally, by adding the values of all these expected cash flows at each payment time t, we can get the expectation of the PVFBP. However, for our long-term disability insurance model studied in this project, we shall derive a more complicated formula for the expectation of the PVFBP since there are both the semiannual disability payments and a lump sum death payment for this case.

Now we start from calculating the first moment of the PVFBP under the cash flow method. Firstly, we define $S_{(x,t,n)}, I_{(x,t,n)}$ and $D_{(x,t,n)}$ as the indicators of being temporarily disabled, permanently disabled and becoming newly deceased (dies since the last status check) at age $x+t$, respectively, for a policyholder who signs an n-year term policy at age x. Define $\mathrm{CF}(x,t,n,FV)$ as the random variable of the cash outflow on the benefit payment at time t for the policyholder who is involved in the policy stated at the beginning of this chapter. Then we can express the cash flow variable in terms of those indicators as

$$CF(x,t,n,FV) = \begin{cases} FV \cdot \left[S_{(x,t,n)} + 2I_{(x,t,n)} + 30D_{(x,t,n)}\right] & t \leq n \\ 0 & t > n \end{cases}, \tag{4.20}$$

for $x, n \in \mathbb{N}^+, t > 0$, where $S_{(x,t,n)}, I_{(x,t,n)}$ and $D_{(x,t,n)}$ follow Bernoulli distribution with parameters $\pi_0(x,t,n), \pi_1(x,t,n), \pi_2(x,t,n)$ and $\pi_3(x,t,n)$, respectively, and $\pi_i(x,t,n)$ for $i = 0,1,2,3$ is the probability of being paid nothing, being temporarily disabled, permanently disabled and newly deceased, respectively, at time t for $t > 0$. The starting values of the probabilities above are given by

$$\pi_i(x,0.5,n) = \begin{cases} Q^{0i}_{(x,1,0.5)}\,, & i = 1,2,3 \\ 1 - \sum\limits_{j=1}^{3} \pi_j(x,0.5,n)\,, & i = 0 \end{cases}. \tag{4.21}$$

It is not difficult to get a recursive formula for $\pi_i(x,t,n)$ for $t > 0.5$ and $i = 1,2,3$; it is given by

$$\pi_i(x,t,n) = \sum_{j=0}^{2} Q^{ji}_{(x+t-0.5,1,0.5)} \pi_j(x,t-0.5,n)\,. \tag{4.22}$$

Therefore, the expected value of the cash outflow at time t for $t > 0.5$ and $i = 1,2,3$ is given by

$$E\left[CF(x,t,n,FV)\right] = FV \cdot E\left[S_{(x,t,n)} + 2I_{(x,t,n)} + 30D_{(x,t,n)}\right] = \sum_{i=1}^{3} b_i \pi_i(x,t,n)\,, \tag{4.23}$$

and the first moment of the PVFBP is calculated as follows:

$$E\left[Z(x,n,FV)\right] = \sum_{t=1}^{2n} E\left[CF(x,0.5t,0.5n,FV)\right] E\left[V(\delta_0,0.5t)\right] \tag{4.24}$$

For the second moment of the PVFBP under the cash flow model, we need to consider the correlation terms between cash flows at different time points (e.g. $Cov\left[CF(x,s,n),CF(x,t,n)\right]$) for $s \neq t$. Parker (1997) introduced a method of calculating the variance of the PVFBP by dividing it into an insurance risk and an investment risk conditioning on rates of return:

$$\begin{aligned}
&Var\left[Z(x,n,FV)\right]\\
=&Var\left\{E\left[Z(x,n,FV) \mid V(\delta_0,k)\right]\right\} + E\left\{Var\left[Z(x,n,FV) \mid V(\delta_0,k)\right]\right\}\\
=&\sum_{s=1}^{2n}\sum_{u=1}^{2n} E\left[V(\delta_0,0.5s)V(\delta_0,0.5u)\right] Cov\left[CF(x,0.5s,0.5n,FV),CF(x,0.5u,0.5n,FV)\right]\\
&+\sum_{s=1}^{2n}\sum_{u=1}^{2n} Cov\left[V(\delta_0,0.5s),V(\delta_0,0.5u)\right] E\left[CF(x,0.5s,0.5n,FV)\right] E\left[CF(x,0.5u,0.5n,FV)\right]
\end{aligned} \tag{4.25}$$

for $x,n \in \mathbb{N}^+$. For the covariance terms, $Cov\left[CF(x,0.5s,n,FV),CF(x,0.5u,n,FV)\right]$, the calculation is more complicated for the long-term disability insurance than that for the traditional life insurance product demonstrated by Parker (1997).

Now rewriting the auto-covariance of cash flows in terms of the covariance terms of these indicators gives:

$$\begin{aligned}
&Cov\left[CF(x,s,n,FV),CF(x,u,n,FV)\right]\\
=&(b_1FV)^2Cov\left[S_{(x,s,n)},S_{(x,u,n)}\right]+(b_2FV)^2Cov\left[I_{(x,s,n)},I_{(x,u,n)}\right]\\
&+(b_3FV)^2Cov\left[D_{(x,s,n)},D_{(x,u,n)}\right]\\
&+b_1b_2FV^2\left\{Cov\left[S_{(x,s,n)},I_{(x,u,n)}\right]+Cov\left[S_{(x,u,n)},I_{(x,s,n)}\right]\right\}\\
&+b_1b_3FV^2\left\{Cov\left[S_{(x,s,n)},D_{(x,u,n)}\right]+Cov\left[S_{(x,u,n)},D_{(x,s,n)}\right]\right\}\\
&+b_2b_3FV^2\left\{Cov\left[I_{(x,s,n)},D_{(x,u,n)}\right]+Cov\left[I_{(x,u,n)},D_{(x,s,n)}\right]\right\}\,,\qquad 0<s\,,u\leq n\,.
\end{aligned}\tag{4.26}$$

To calculate the covariance terms of these indicators, we need to discuss two situations: $s=u$ and $s\neq u$, below.

1. If $s=u$, then we have

$$\begin{cases}
Cov\left[D_{(x,s,n)},D_{(x,u,n)}\right]=\pi_3(x,s,n)\left[1-\pi_3(x,s,n)\right]\\
Cov\left[S_{(x,s,n)},S_{(x,u,n)}\right]=\pi_1(x,s,n)\left[1-\pi_1(x,s,n)\right]\\
Cov\left[I_{(x,s,n)},I_{(x,u,n)}\right]=\pi_2(x,s,n)\left[1-\pi_2(x,s,n)\right]\\
Cov\left[S_{(x,s,n)},D_{(x,u,n)}\right]=Cov\left[S_{(x,u,n)},D_{(x,s,n)}\right]=-\pi_1(x,s,n)\pi_3(x,s,n)\\
Cov\left[I_{(x,s,n)},D_{(x,u,n)}\right]=Cov\left[I_{(x,u,n)},D_{(x,s,n)}\right]=-\pi_2(x,s,n)\pi_3(x,s,n)\\
Cov\left[I_{(x,s,n)},S_{(x,u,n)}\right]=Cov\left[I_{(x,u,n)},S_{(x,s,n)}\right]=-\pi_1(x,s,n)\pi_2(x,s,n)
\end{cases}\,,\tag{4.27}$$

   for $0<s=u\leq n$ and for $s=u\geq n$,

$$\begin{aligned}
Cov\left[D_{(x,s,n)},D_{(x,u,n)}\right]&=Cov\left[S_{(x,s,n)},S_{(x,u,n)}\right]=Cov\left[I_{(x,s,n)},I_{(x,u,n)}\right]\\
&=Cov\left[S_{(x,s,n)},D_{(x,u,n)}\right]=Cov\left[I_{(x,s,n)},D_{(x,u,n)}\right]\\
&=Cov\left[I_{(x,s,n)},S_{(x,u,n)}\right]=0\,.
\end{aligned}$$

2. If $0<s<u$ (we do not need to discuss the situation that $s>u$ owing to symmetry), we have for $s<u\leq n$,

$$\begin{aligned}
Cov\left[D_{(x,s,n)},D_{(x,u,n)}\right]&=-E\left[D_{(x,s,n)}\right]E\left[D_{(x,u,n)}\right]=-\pi_3(x,s,n)\pi_3(x,u,n)\,,\\
Cov\left[D_{(x,s,n)},S_{(x,u,n)}\right]&=-E\left[D_{(x,s,n)}\right]E\left[S_{(x,u,n)}\right]=-\pi_3(x,s,n)\pi_1(x,u,n)\,,\\
Cov\left[D_{(x,s,n)},I_{(x,u,n)}\right]&=-E\left[D_{(x,s,n)}\right]E\left[I_{(x,u,n)}\right]=-\pi_3(x,s,n)\pi_2(x,u,n)\,,
\end{aligned}\tag{4.28}$$

$$
\begin{aligned}
Cov\left[I_{(x,s,n)}, D_{(x,u,n)}\right] &= E\left[I_{(x,s,n)}D_{(x,u,n)}\right] - E\left[I_{(x,s,n)}\right]E\left[D_{(x,u,n)}\right] \\
&= \pi_2(x,s,n)Q^{23}_{(x+s,2u-2s,0.5)} - \pi_2(x,s,n)\pi_3(x,u,n)\,, \\
Cov\left[I_{(x,s,n)}, I_{(x,u,n)}\right] &= E\left[I_{(x,s,n)}I_{(x,u,n)}\right] - E\left[I_{(x,s,n)}\right]E\left[I_{(x,u,n)}\right] \\
&= \pi_2(x,s,n)Q^{22}_{(x+s,2u-2s,0.5)} - \pi_2(x,s,n)\pi_2(x,u,n)\,, \\
Cov\left[I_{(x,s,n)}, S_{(x,u,n)}\right] &= E\left[I_{(x,s,n)}S_{(x,u,n)}\right] - E\left[I_{(x,s,n)}\right]E\left[S_{(x,u,n)}\right] \\
&= -\pi_2(x,s,n)\pi_1(x,u,n)\,, \qquad (4.29)
\end{aligned}
$$

and

$$
\begin{aligned}
Cov\left[S_{(x,s,n)}, D_{(x,u,n)}\right] &= E\left[S_{(x,s,n)}D_{(x,u,n)}\right] - E\left[S_{(x,s,n)}\right]E\left[D_{(x,u,n)}\right] \\
&= \pi_1(x,s,n)Q^{13}_{(x+s,2u-2s,0.5)} - \pi_1(x,s,n)\pi_3(x,u,n)\,, \\
Cov\left[S_{(x,s,n)}, I_{(x,u,n)}\right] &= E\left[S_{(x,s,n)}I_{(x,u,n)}\right] - E\left[S_{(x,s,n)}\right]E\left[I_{(x,u,n)}\right]) \\
&= \pi_1(x,s,n)Q^{12}_{(x+s,2u-2s,0.5)} - \pi_1(x,s,n)\pi_2(x,u,n)\,, \\
Cov\left[S_{(x,s,n)}, S_{(x,u,n)}\right] &= E\left[S_{(x,s,n)}S_{(x,u,n)}\right] - E\left[S_{(x,s,n)}\right]E\left[S_{(x,u,n)}\right] \\
&= \pi_1(x,s,n)Q^{11}_{(x+s,2u-2s,0.5)} - \pi_1(x,s,n)\pi_1(x,u,n)\,. \qquad (4.30)
\end{aligned}
$$

Otherwise, if $u > n$, we have

$$
\begin{aligned}
Cov\left[D_{(x,s,n)}, D_{(x,u,n)}\right] &= Cov\left[S_{(x,s,n)}, S_{(x,u,n)}\right] = Cov\left[I_{(x,s,n)}, I_{(x,u,n)}\right] \\
&= Cov\left[S_{(x,s,n)}, D_{(x,u,n)}\right] = Cov\left[I_{(x,s,n)}, D_{(x,u,n)}\right] \\
&= Cov\left[I_{(x,s,n)}, S_{(x,u,n)}\right] = 0
\end{aligned}
$$

Finally, by applying (4.25) - (4.30), it is not difficult to get the variance of the PVFBP using the cash flow method. In general, there are several advantages of the cash flow method compared to the recursive formula:

1. The cash flow method is much less complicated than the recursive formula shown in previous section.
2. The cash flow method is more efficient than the recursive method in programming. For example, the moment calculation process takes several hours by applying the recursive method but only costs less than 1 second using the cash flow method.
3. By applying the cash flow method, we are able to classify the risks and look at the portion of investment risk and insurance risk respectively in order to make up a good hedging strategy.

In the following chapter, we shall introduce the risk analysis of an insurance portfolio under the cash flow method.

Chapter 5

Valuation of Long-term Disability Insurance Portfolio

In this chapter, we shall extend the result obtained for a single policy to the insurance portfolios which consists of great many numbers of policies. In addition, we shall also calculate the insurance risk and the investment risk of the insurance portfolio by applying the cash flow method only.

5.1 Cash Flow Method for Homogeneous Portfolio

In this section, we study the homogeneous portfolio case, that is, all the policies in this insurance portfolio have the same face value. It implies that all the policies in this insurance portfolio have the same face value and all the policyholders included join the plan at the same age.

To calculate the moments of the PVFBP for this insurance portfolio, we need to follow the steps as was done in Chapter 4 by deriving the expected value, variance and auto-covariance terms of the total cash outflow at each payment time. Define $CF(x, t, n, FV, c)$ the total cash flow paid out at time t for a homogeneous portfolio which consists of c n-year term policies with all the policyholders aged x at the time of entry and the temporary disability benefit payment to be FV. Let x_i be the age of the ith policyholder ($x_i = x$ for all i in this case); then the random variable of cash flow at time t can be expressed as

$$CF(x,t,n,FV,c) = \sum_{i=1}^{c} FV \left[S_{(x_i,t,n)} + 2I_{(x_i,t,n)} + 30D_{(x_i,t,n)}\right] \tag{5.1}$$

for $c, x \in \mathbb{N}^+$ and $FV, t, n > 0$.

Therefore, under the assumption that each policyholder insured in the insurance portfolio has an identical and independent mortality distribution, the expected value of the cash outflow at time t for the whole homogeneous portfolio can be expressed as

$$\begin{aligned} E\left[CF(x,t,n,FV,c)\right] &= E\left[\sum_{i=1}^{c} CF(x_i,t,n,FV)\right] \\ &= cE\left[CF(x_i,t,n,FV)\right] \\ &= c\sum_{i=1}^{3} b_i \pi_i(x,t,n)\,, \end{aligned} \tag{5.2}$$

where $b_1 = FV, b_2 = 2FV$, and $b_3 = 30FV$. Similarly, it is not difficult to extend the results in (4.25) - (4.29) to get the covariance terms of the portfolio:

$$\begin{aligned} &Cov\left[CF(x,s,n,FV,c), CF(x,u,n,FV,c)\right] \\ &= Cov\left[\sum_{i=1}^{c} CF(x_i,s,n,FV), \sum_{i=1}^{c} CF(x_i,u,n,FV)\right] \\ &= \sum_{i=1}^{c}\sum_{j=1}^{c} Cov\left[CF(x_i,s,n,FV), CF(x_j,u,n,FV)\right] \\ &= \sum_{i=1}^{c} Cov\left[CF(x_i,s,n,FV), CF(x_i,u,n,FV)\right] \\ &= c\,Cov\left[CF(x,s,n,FV), CF(x,u,n,FV)\right] \end{aligned} \tag{5.3}$$

for $c, x \in \mathbb{N}^+$ and $s, u, n, FV > 0$ since the transition process for each policyholder is uncorrelated.

5.2 Cash Flow Method for Non-homogeneous Portfolio

In this section, we consider a more general case, namely the non-homogeneous portfolio case, where the start age of the policyholder included, the face value of each policy, and the length of the policies are not identical. The method of approaching the expectation and the autocovariance terms of the total cash outflow is similar to the one for the homogeneous portfolio. The idea is to classify the whole portfolio into several small homogeneous portfolios.

Now assume that a non-homogeneous portfolio which consists of m homogeneous portfolios and the ith n_i-year term insurance portfolio consists of c_i policies with the same entry age x_i and the face value, FV_i, where $\sum_{i=1}^{m} c_i = N$.

Furthermore, we define the following sets:

$$
\begin{cases}
\chi = \{x_1, x_2, \ldots, x_m\} \\
\eta = \{n_1, n_2, \ldots, n_m\} \\
\digamma = \{FV_1, FV_2, \ldots, FV_m\} \\
\zeta = \{c_1, c_2, \ldots, c_m\}
\end{cases}
$$

to be the information sets indicating ages, terms, face values and numbers of policies, respectively, for the non-homogeneous portfolio. Then we are able to define the cash outflow at time t for a non-homogeneous portfolio stated above in terms of the information sets, $CF(\chi, t, \eta, \digamma, \zeta, m)$.

Then the expected value of such non-homogeneous portfolio at time t can be obtained by extending the results for the single policy:

$$
\begin{aligned}
E\left[CF(\chi, t, \eta, \digamma, \zeta, m)\right] &= \sum_{i=1}^{m} E\left[CF(x_i, t, n_i, FV_i, c_i)\right] \\
&= \sum_{i=1}^{m} c_i E\left[CF(x_i, t, n_i, FV_i)\right] \,, \qquad x_i, c_i \in \mathbb{N}^+ \,, FV_i \,, t \,, n_i > 0 \,.
\end{aligned}
\tag{5.4}
$$

Similarly, since the fact that the transition process of any policyholder(s) among homogeneous group(s) is uncorrelated still applies, we have the following covariance term:

$$
\begin{aligned}
&Cov\left[CF(\chi, s, \eta, F, \zeta, m), CF(\chi, u, \eta, F, \zeta, m)\right] \\
&= Cov\left[\sum_{i=1}^{m}\sum_{j=1}^{c_i} CF(x_i, s, n_i, FV_i), \sum_{i=1}^{m}\sum_{j=1}^{c_i} CF(x_i, u, n_i, FV_i)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{c_i} Cov\left[CF(x_i, s, n_i, FV_i), CF(x_i, u, n_i, FV_i)\right] \\
&= \sum_{i=1}^{m} c_i Cov\left[CF(x_i, s, n_i, FV_i), CF(x_i, u, n_i, FV_i)\right]\,, \qquad x_i, c_i \in \mathbb{N}^+\,, FV_i\,, s\,, u\,,\,, n_i > 0\,.
\end{aligned}
\tag{5.5}
$$

With all the closed forms of the expectation, variance and auto-covariance functions of the cash flow, the next step is to analyze the insurance risk and investment risk of the whole non-homogeneous portfolio.

5.3 Risk Analysis of Non-homogeneous Portfolio

Parker (1997) introduced two ways to classify the insurance and investment risks under the cash flow method for an insurance portfolio of term life and endowment contracts by conditioning on the interest rates and the mortalities, respectively. For our disability insurance model, we apply a similar method to analyze the risks. Note that in our case, the mortality distribution is replaced by the multi-state transition process.

Now we introduce the first way of dividing the total riskiness per policyholder involved in the insurance portfolio, which is denoted by

$$
Var\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i, n_i, FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\,, \qquad x_i\,, c_i \in \mathbb{N}^+\,, n_i\,, FV_i > 0\,. \tag{5.6}
$$

1. The first way is to divide the risk by conditioning on the interest rate process, denoted

by $I = \{\delta_k; k \in \mathbb{N}\}$

$$
\begin{aligned}
& Var\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i, n_i, FV_i)}{\sum\limits_{i=1}^{m} c_i}\right) \\
=& E\left\{Var\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i, n_i, FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg| I\right]\right\} \\
& + Var\left\{E\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i, n_i, FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg| I\}\right]\right\},
\end{aligned} \tag{5.7}
$$

where the first term above corresponds to the so-called insurance risk, which can be further calculated by

$$
\begin{aligned}
& E\left\{Var\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i, n_i, FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg| I\right]\right\} \\
=& E\left\{Var\left[\frac{\sum\limits_{s=1}^{2n} V(\delta_0, 0.5s) CF(\chi, 0.5s, \eta, \digamma, \zeta, m)}{\sum\limits_{i=1}^{m} c_i}\Bigg| I\right]\right\} \\
=& \frac{1}{\left(\sum\limits_{i=1}^{m} c_i\right)^2} E\Bigg\{\sum_{s=1}^{2n}\sum_{u=1}^{2n}\Bigg[V(\delta_0, 0.5s)V(\delta_0, 0.5u) \\
& \qquad \times Cov\left[CF(\chi, 0.5s, \eta, \digamma, \zeta, m), CF(\chi, 0.5u, \eta, \digamma, \zeta, m)\right]\Bigg]\Bigg\} \\
=& \frac{1}{\left(\sum\limits_{i=1}^{m} c_i\right)^2} \sum_{s=1}^{2n}\sum_{u=1}^{2n}\sum_{i=1}^{m}\Bigg\{c_i E\left[V(\delta_0, 0.5s)V(\delta_0, 0.5u)\right] \\
& \qquad \times Cov\left[CF(x_i, 0.5s, n_i, FV_i), CF(x_i, 0.5u, n_i, FV_i)\right]\Bigg\}
\end{aligned} \tag{5.8}
$$

for $m, x_i, c_i \in \mathbb{N}^+, n_i, FV_i > 0, n = max\{n_i\}$ and the second term standing for the investment risk can be further calculated by:

$$
\begin{aligned}
&Var\left\{E\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i,n_i,FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg| I\right]\right\}\\
=&Var\left\{E\left[\frac{\sum\limits_{s=1}^{2n} V(\delta_0,0.5s)CF(\chi,0.5s,\eta,\digamma,\zeta,m)}{\sum\limits_{i=1}^{m} c_i}\Bigg| I\right]\right\}\\
=&Var\left\{\frac{\sum\limits_{s=1}^{2n}\sum\limits_{i=1}^{m} c_i E\left[CF(x_i,0.5s,n_i,FV_i)\right]V(\delta_0,0.5s)}{\sum\limits_{i=1}^{m} c_i}\right\}\\
=&\frac{1}{\left(\sum\limits_{i=1}^{m} c_i\right)^2}\sum_{s=1}^{2n}\sum_{u=1}^{2n}\sum_{i=1}^{m} c_i^2\Bigg\{E\left[CF(x_i,0.5s,n_i,FV_i)\right]E\left[CF(x_i,0.5u,n_i,FV_i)\right]\\
&\qquad\qquad\times Cov\left[V(\delta_0,0.5s),V(\delta_0,0.5u)\right]\Bigg\}
\end{aligned}
\tag{5.9}
$$

for $x_i, c_i \in \mathbb{N}^+, n_i, FV_i > 0$.

2. An alternative way is to divide the total risk by conditioning on the transition process of each policyholder included in the insurance portfolio, where the transition process of the policyholders in the portfolio is denoted by M. The total risk is divided as follows:

$$
\begin{aligned}
&Var\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i,n_i,FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\\
=&E\left\{Var\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i,n_i,FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg| M\right]\right\}\\
&+Var\left\{E\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i,n_i,FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg| M\right]\right\}
\end{aligned}
\tag{5.10}
$$

for $x_i, c_i \in \mathbb{N}^+$ and $FV_i, n_i > 0$. The first term standing for the investment risk can be calculated by

$$
\begin{aligned}
&E\left\{Var\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i,n_i,FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg|M\right]\right\}\\
=&E\left\{Var\left[\frac{\sum\limits_{s=1}^{2n} CF(\chi,0.5s,\eta,\digamma,\zeta,m)V(\delta_0,0.5s)}{\sum\limits_{i=1}^{m} c_i}\Bigg|M\right]\right\}\\
=&\frac{1}{\left(\sum\limits_{i=1}^{m} c_i\right)^2}\sum_{s=1}^{2n}\sum_{u=1}^{2n}\Bigg\{E\left[CF(\chi,0.5s,\eta,\digamma,\zeta,m)CF(\chi,0.5u,\eta,\digamma,\zeta,m)\right]\\
&\qquad\qquad\times Cov\left[V(\delta_0,0.5s),V(\delta_0,0.5u)\right]\Bigg\}
\end{aligned}
\tag{5.11}
$$

while the second term representing the insurance risk is

$$
\begin{aligned}
&Var\left\{E\left[\left(\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{c_i} Z_i(x_i,n_i,FV_i)}{\sum\limits_{i=1}^{m} c_i}\right)\Bigg|M\right]\right\}\\
=&Var\left\{E\left[\frac{\sum\limits_{s=1}^{2n} CF(\chi,0.5s,\eta,\digamma,\zeta,m)V(\delta_0,0.5s)}{\sum\limits_{i=1}^{m} c_i}\Bigg|M\right]\right\}\\
=&Var\left\{\frac{\sum\limits_{s=1}^{2n}\sum\limits_{i=1}^{m} c_iCF(x_i,t,n_i,FV_i)E\left[V(\delta_0,0.5s)\right]}{\sum\limits_{i=1}^{m} c_i}\right\}\\
=&\frac{1}{\left(\sum\limits_{i=1}^{m} c_i\right)^2}\sum_{s=1}^{2n}\sum_{u=1}^{2n}\sum_{i=1}^{m}\Bigg\{c_iE\left[V(\delta_0,0.5s)\right]E\left[V(\delta_0,0.5u)\right]\\
&\qquad\qquad\times Cov\left[CF(x_i,0.5s,n_i,FV_i),CF(x_i,0.5u,n_i,FV_i)\right]\Bigg\}
\end{aligned}
\tag{5.12}
$$

for $x_i, c_i \in \mathbb{N}^+$, $FV_i, n_i > 0$.

By applying the results derived in (5.4) - (5.12), we are able to analyze numerically the risks for a non-homogeneous insurance portfolio. In Chapter 6, we shall analyze the numerical valuation results of a general long-term disability insurance portfolio.

Chapter 6

Numerical Illustration

In this chapter, we illustrate the numerical moment calculation results for the single policy and the general insurance portfolios. The insurance risk and the investment risk across age groups upon the term of the policy are studied. For the single policy case, sensitivity tests have been done for several parameters of the long-term disability insurance models such as the benefit payments ratios and the convergence speed of the AR(1) interest rate process. For the portfolio case, the insurance risk and the investment risk with three different interest rate models are analyzed. Cash flow of future benefit payments are calculated. Extreme cases are considered under the binomial tree interest rate model.

6.1 Single Policy Case

We consider an n-year term disability policy introduced in the previous chapters with semi-annual disability payments of \$1 for temporarily disability (TD Benefit), \$2 for permanent disability (PD Benefit) and a lump sum death benefit of \$30 to policyholders at working ages. For the single policy case, the expectation and the variance of the PVFBP, which can be used for premium calculations and risk analysis, are calculated in this section.

The three interest rate models presented in Chapter 3 are used for the calculation process: the deterministic interest model, the binomial tree model and the AR(1) process. For the base scenario, the values of the parameters in the three models are set up as follows. Note that in the following we sometimes refer to the expectation and the variance as the first two moments.

1. For the deterministic interest rate model, we assume the annual constant interest rate is $\delta_k = 0.06\,, k \in \mathbb{N}$.

2. For the binomial tree interest rate model, we assume that the starting value of the annual interest rate and the risk-free interest rate for a zero-coupon bond are $\delta_0 = r = 0.06$. This is a special case which has the same expectation of the discounted interest rates, $E[V(\delta_0, k)]\,, k \in \mathbb{N}$, as the one in the deterministic interest rate model. The expectation and the variance of the interest rate functions are estimated by simulations for 1 million times.

3. For the AR(1) model, we assume that the long-term mean and the starting value of the annual interest rate is $\delta_0 = 0.06$ and the long-term mean of the annual force of interest is $ln(1 + \delta_0) = ln(1.06)$. The convergence speed to this long-term mean is $\phi = 0.9$ with the volatility of the residual being $\sigma_a = 0.01$.

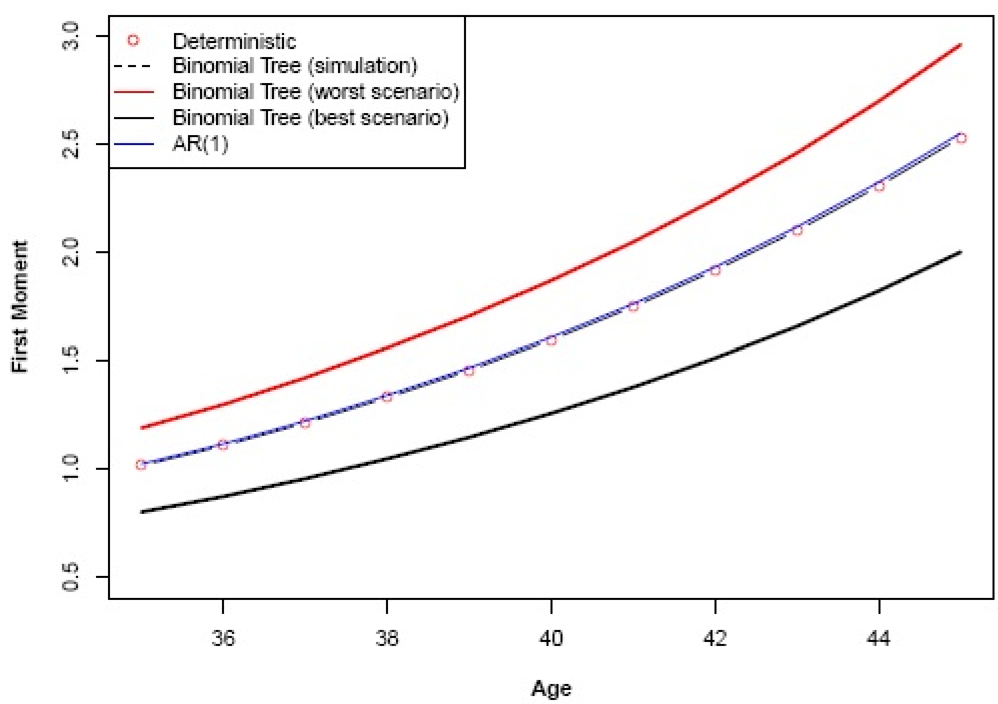

Figure 6.1: Expectation of PVFBP vs Age for a 15-year Disability Insurance Policy with Three Interest Rate Models under The Base Scenario

We first calculate the moments with the three interest rate models. Note that for single policy case, the numerical results under the cash flow method and the recursive method are exactly the same. Therefore, we illustrate the results for both the single policy and the portfolio case under the cash flow method for efficiency purpose.

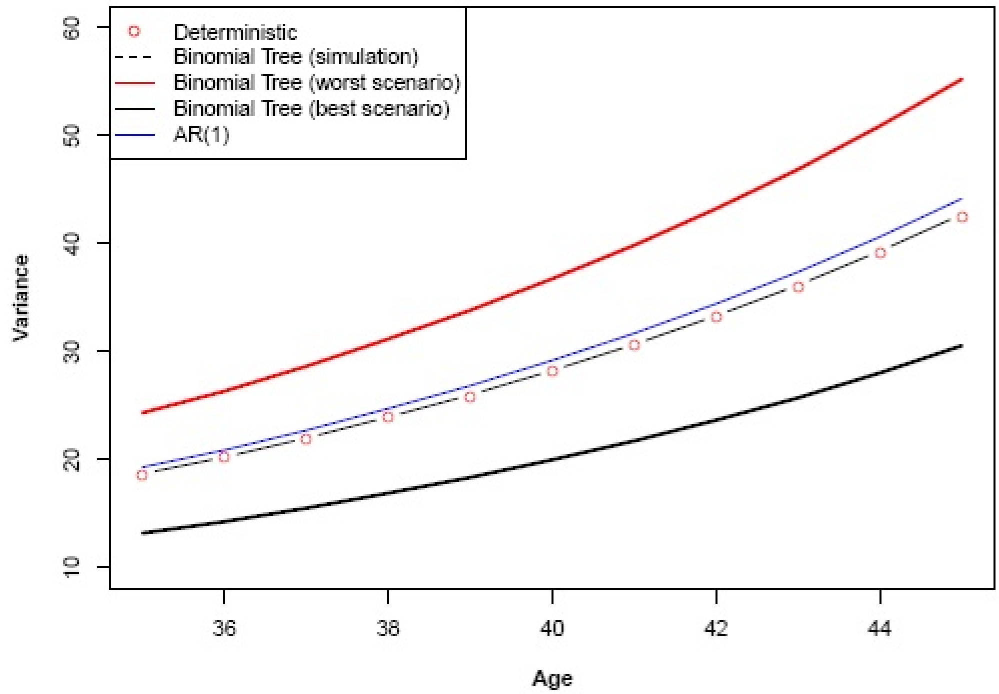

Figure 6.2: Variance of PVFBP vs Age for a 15-year Disability Insurance Policy with Three Interest Rate Models

Figures 6.1 and 6.2 show the numerical results of the expected value and variance of the present value of future payments of 15-year long-term disability contracts for insureds who entered the plan between age 35 and 45, respectively. Obviously, for the single policy case, the numerical results under the three interest rate models mentioned above are highly consistent (the values almost fall on the same line and the binomial interest model is equivalent to the deterministic model in this case). For the binomial tree interest rate model, the best scenario refers to the lowest annual interest rates in each year being selected to form the upper bound of the PVFBP and the worst scenario is the case where the highest annual interest rates being selected (see the two lines in bold) to form the lower bound of the PVFBP. For the age groups of 35-45, the mortality rate is rapidly growing at senior ages. Therefore, the first moment of the PVFBP has a concave up and increasing trend with respect to the age since the death benefit payment is a major payment under our long-term disability policy. The variance of the PVFBP which can be considered as a rough estimate of the insurance risk for the single policy case (when investment risk is relatively small) is also increasing in age for the ages considered, 35-55.

The moment calculation results are very close under the three interest rate models, especially the binomial tree model and the deterministic model (the first moments are the same). We will show further graphical results for moment calculations and the sensitivity tests of the benefit payment amounts under the AR(1) model only. Figure 6.3 shows that the first moment and the variance of the PVFBP across the entry age group 35-45 with terms of policies from 1 to 15 years are increasing functions in both the entry ages and the terms of policy. By comparing the growth of the mortality rates within each age intervals, we observe that policyholders from 38-39, 40-41, 42-43, 46-47, 49-50 and 52-53 experience relatively rapid growth in the mortality rates. This implies that for a 1-year term policy, the growth of the first two moments as a function of age is relatively high in the age intervals mentioned above. For example, mathematically we have

$$E[Z(39,1,1)] - E[Z(38,1,1)] > E[Z(38,1,1)] - E[Z(37,1,1)]$$

since age group 38-39 is a peak of the growth in mortality rates. However, the longer the term of the policy is, the less impact the differences in the growth of mortality rates in the last year of the policy have on the first two moments of the PVFBP. The first two moments tend to be concave up in age for long-term policies.

Figure 6.4 shows both the insurance risk (left) and the investment risk (right) for a 15-year disability insurance policy with the AR(1) interest rate process. The insurance risk observed is concave up in age and the term of policies. The investment risk for terms of policies longer than 2 years, though relatively small compared to the insurance risk, has a more obvious concave up trend in both the age and the term of policy than the insurance risk does. In general, the longer the term of the policy (or the more senior the age) is, the faster the investment risk is increasing. To calculate the investment risk and the insurance risk in Figure 6.4, we use (5.9) and (5.12) since Parker (1997) suggested that these two formulas provide better estimates than the methods expressed by (5.8) and (5.11).

In addition to the studies under the base scenario, the next step is to do some sensitivity

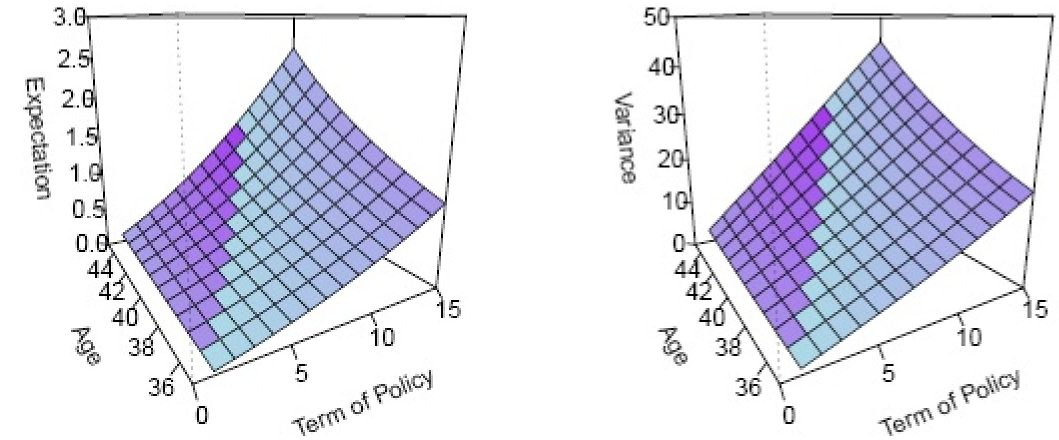

Figure 6.3: Expectation and Variance of PVFBP vs Term Policy and Age under the AR(1) Interest Rate Model ($\phi = 0.9; \sigma_a = 0.01; \delta_0 = 0.06$)

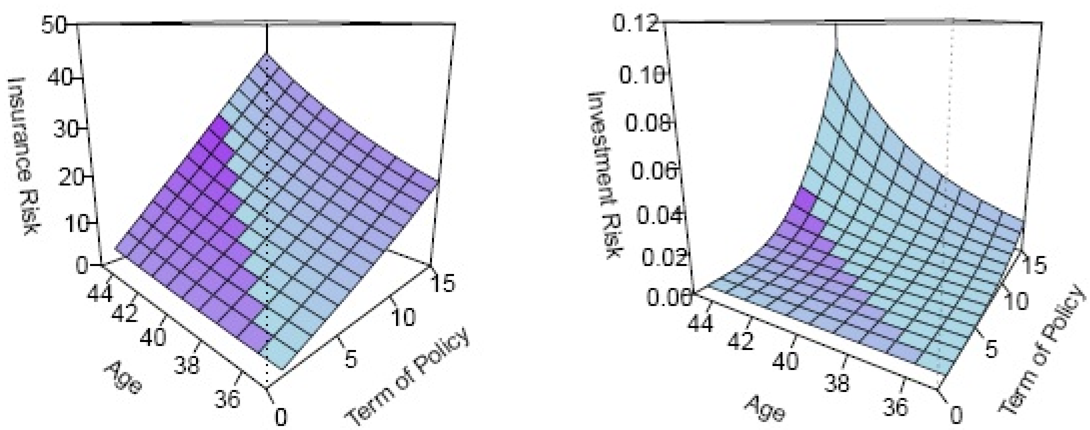

Figure 6.4: Insurance and Investment Risk of PVFBP vs Term Policy and Age under the AR(1) Interest Rate Model ($\phi = 0.9; \sigma_a = 0.01; \delta_0 = 0.06$)

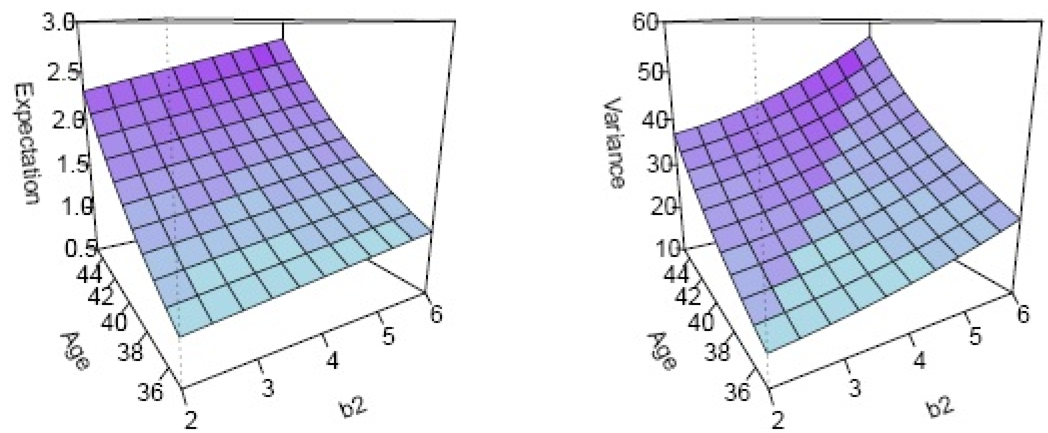

Figure 6.5: Expectation and Variance of PVFBP vs Disability Benefit b_2 for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($b_1 = 1; b_3 = 30; \phi = 0.9; \sigma_a = 0.01; \delta_0 = 0.06$)

tests on the benefit payment amounts and the parameters in the interest rate model. Figures 6.5 and 6.6 illustrate the benefit ratios within two disability payment levels and the ratios between death benefit and semiannual disability payments, respectively. Since the numerical results under three different interest rate models differ not much, here we show only the numerical results under the AR(1) interest rate process for 15-year term policies in which policyholders entered the plan between ages 35 and 45.

In Figure 6.5, we have fixed the benefit for temporarily disabled, $b_1 = 1$, and the death benefit, $b_3 = 30$, to test the impact of changing in permanent disability benefit payment b_2. While Figure 6.6, $b_1 = 1$ and $b_2 = 2$ are fixed and we test the impact of changing in death benefit payment b_3.

Since two factors have been fixed for the sensitivity tests, changing the third factor is equivalent to adding a certain amount of supplementary benefit for the permanent disability (the death). Therefore, the change in the expectation of PVFBP, $Z(x, 15, 1)$, is proportional to the change in the benefit amount b_2 (b_3), Δb_2 (Δb_3). The change in the expectation of PVFBP with respect to the change in the benefit amount, $\frac{\Delta E[Z(x,15,1)]}{\Delta b_2}$ ($\frac{\Delta E[Z(x,15,1)]}{\Delta b_3}$), is higher for the senior age groups owing to the larger possibility of paying the supplementary benefits.

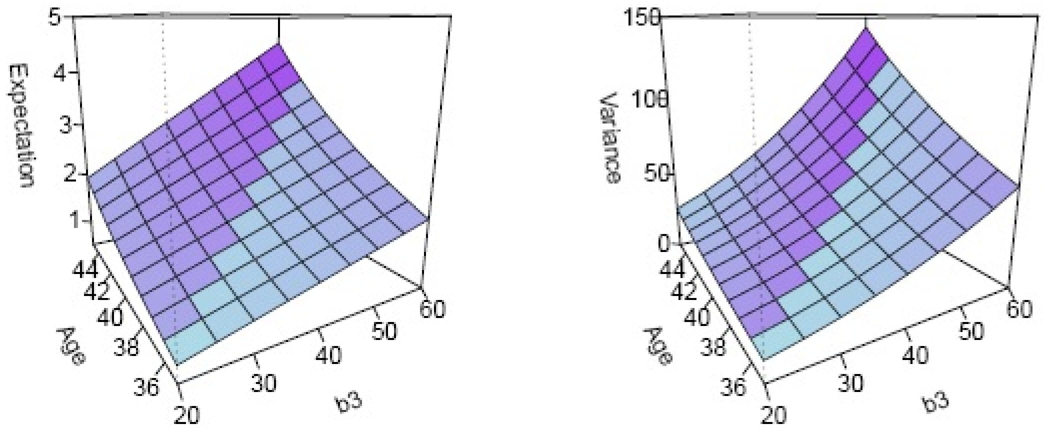

Figure 6.6: Expectation and Variance of PVFBP vs Death Benefit b_3 for a 15-year Disability Insurance Policy under the AR(1) Interest Models ($b_1 = 1; b_2 = 2; \phi = 0.9; \sigma_a = 0.01; \delta_0 = 0.06$)

In addition, the second moment of PVFBP, $E[Z^2(x, 15, 1)]$, includes more terms which are related to the benefit amount b_2 (b_3). Therefore, the larger b_2 (b_3) is, the more significant its impact has on the dollar amount change in $E[Z^2(x, 15, 1)]$. We discover that there is a concave up trend of the variance terms in the benefit ratios, especially when the changing factor is the major benefit payments (i.e. b_3) in our policy (see Figure 6.6). In other words, the variance of the PVFBP is more sensitive for higher benefit ratios. Again, a combined effect of a larger possibility of the supplementary payments for the senior aged groups and the large value of the major benefit b_3, the variance terms of the PVFBP is concave up in age. However, if we fix the death benefit amount and change the permanent disability benefit amount which is relatively small compared to the former one, we do not necessarily have a concave curve (see Figure 6.5).

Now we perform the sensitivity tests for the model parameters in the AR(1) interest rate model by starting from ϕ, the convergence speed to the long-term mean of the annual interest rate. From Figure 6.7, we can conclude that the expected value of the PVFBP has a concave up shape in both the age and the convergence speed, ϕ, though the increase in the changing speed in ϕ is small. Similarly, we observe exactly the same principle in the insurance risk. In other words, the expected value of the PVFBP and the insurance risk of

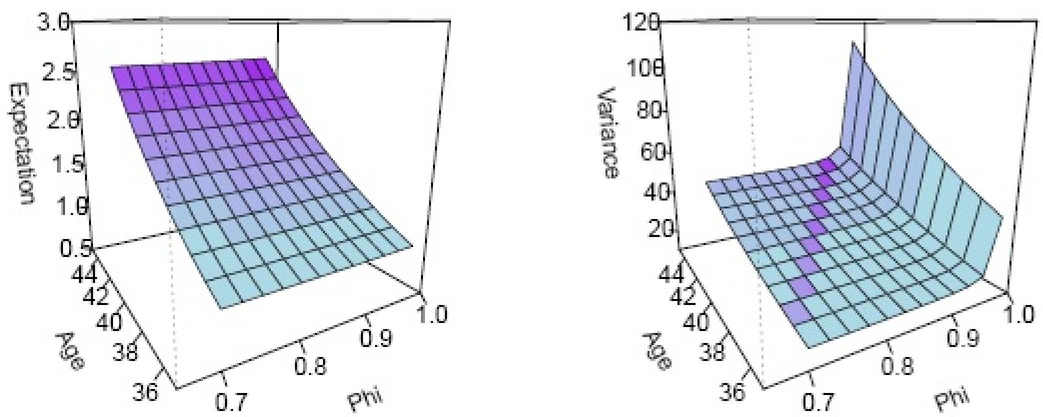

Figure 6.7: Expectation and Variance of PVFBP vs ϕ and Age for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($\sigma_a = 0.01; \delta_0 = 0.06$)

a 15-year term policy is more sensitive to the changes in age than that of the convergence speed, ϕ, in the AR(1) interest rate model.

Furthermore, by looking at the investment risk shown in Figure 6.8 (right), it is not surprising to see the sudden change in the variance terms when the convergence speed is high. Note that the insurance risk and the investment risk shown here do not add up to the variance term shown in Figure 6.7, since we are using two different approaches to get the best estimates of the risks (see (5.9) and (5.12) for reference).

The investment risk is extremely sensitive when ϕ is close to 1. Theoretically, the intermediate term, $a(s, u)$, in calculating the auto-covariance terms of the interest rate functions turns huge when $1 - \phi$ is close to zero (see Proposition 3.5 for reference). This explains the reason why the investment risk has a turn point when ϕ is close to 1.

Figures 6.9 and 6.10 illustrate the sensitivity test results on δ_0, the long-term mean parameter of the annual interest rate in the AR(1) interest rate model. Overall, the expectation, the variance of the PVFBP, the insurance risk and the investment risk are all decreasing in the long-term mean parameter. Their concave up and decreasing trends in age are still satisfied at any values of the long-term mean parameter δ_0 and the gwoth is fairly slow at large values of δ_0. The concave down shape of the change of the four quantities in δ_0 can also be observed from Figures 6.9 and 6.10. To sum up, the four quantities are

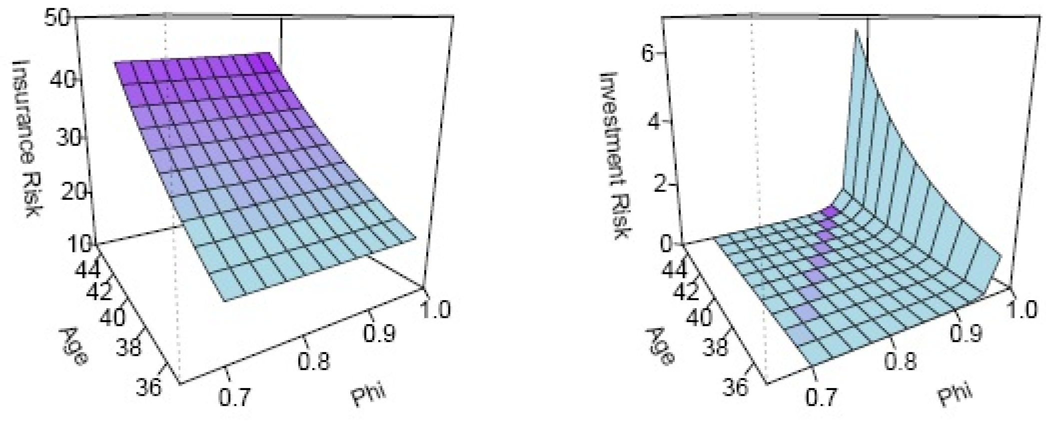

Figure 6.8: Insurance and Investment Risk vs ϕ and Age for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($\sigma_a = 0.01; \delta_0 = 0.06$)

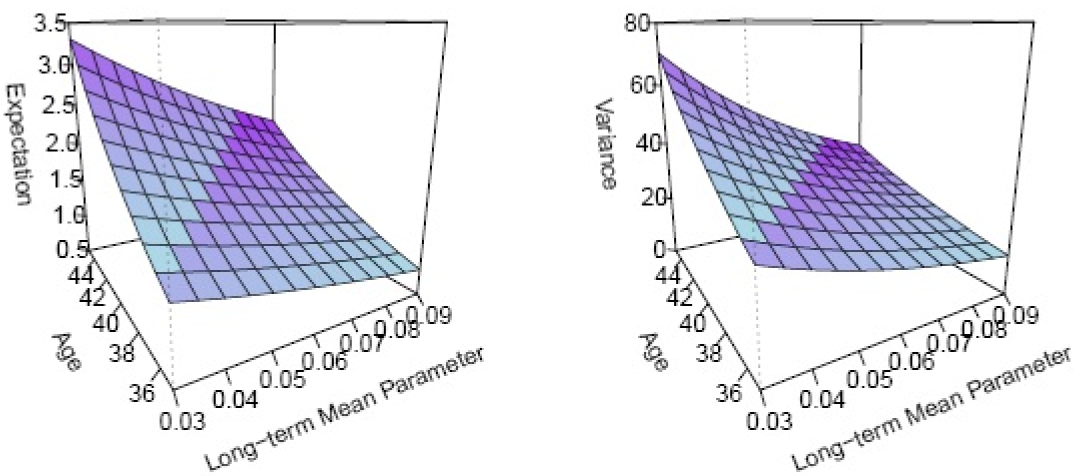

Figure 6.9: Expectation and Variance of PVFBP vs δ_0 and Age for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($\sigma_a = 0.01; \phi = 0.9$)

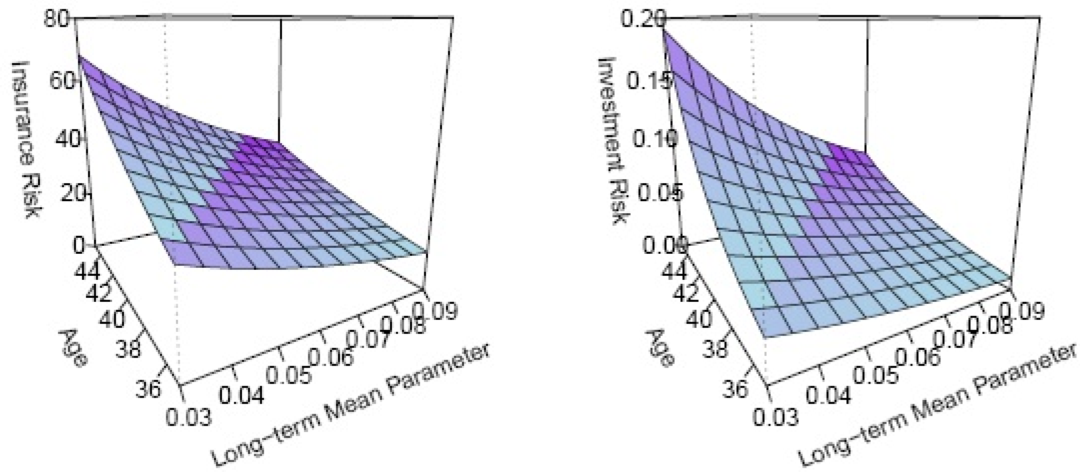

Figure 6.10: Insurance and Investment Risk vs δ_0 and Age for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($\sigma_a = 0.01; \phi = 0.9$)

decreasing functions of δ_0, and are more sensitive to small values of δ_0.

Figures 6.11 and 6.12 show the numerical results of the sensitivity test due to the change in volatility term in the AR(1) interest rate model, σ_a. Not surprisingly by its name, the investment risk is increasing in σ_a with a quicker speed at large values of σ_a and senior ages (see Figure 6.12). This is because a more senior age gives rise to a larger expected values on the cash flow on the benefit payment. Furthermore, the expectation, variance and auto-covariance terms of the interest rate functions which are used to calculate the insurance risk are increasing functions of σ_a^2 (see Chapter 3).

In summary, all the four terms are concave up functions of σ_a, and they are more sensitive to the change in σ_a at the senior age groups. While the investment risk is the most sensitive one to the change in σ_a, the sensitivity of the variance of the PVFBP lies between that of the investment risk and the insurance risk by combining their effects.

6.2 A Numerical Example of Non-homogeneous Portfolio

In this section, we illustrate the numerical results for portfolio cases under the three interest rate models mentioned in Section 6.1. The total riskiness of a non-homogeneous long-term

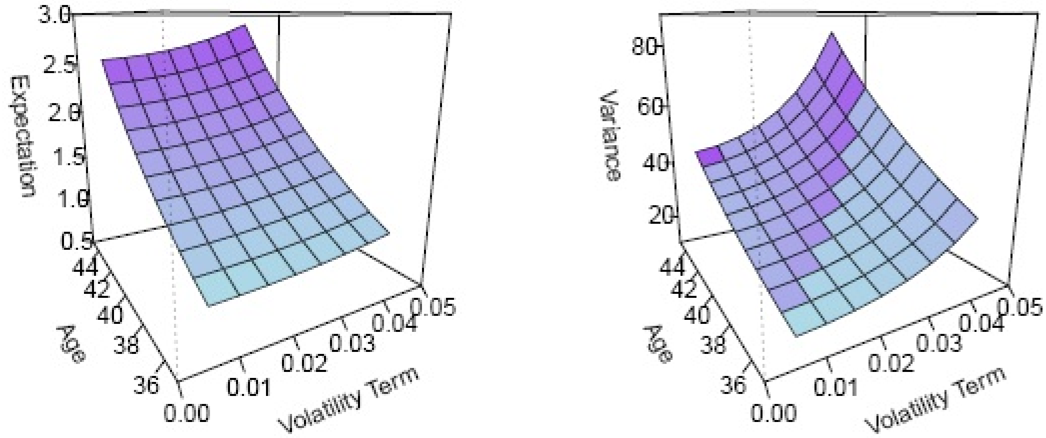

Figure 6.11: Expectation and Variance of PVFBP vs σ_a and Age for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($\delta_0 = 0.06; \phi = 0.9$)

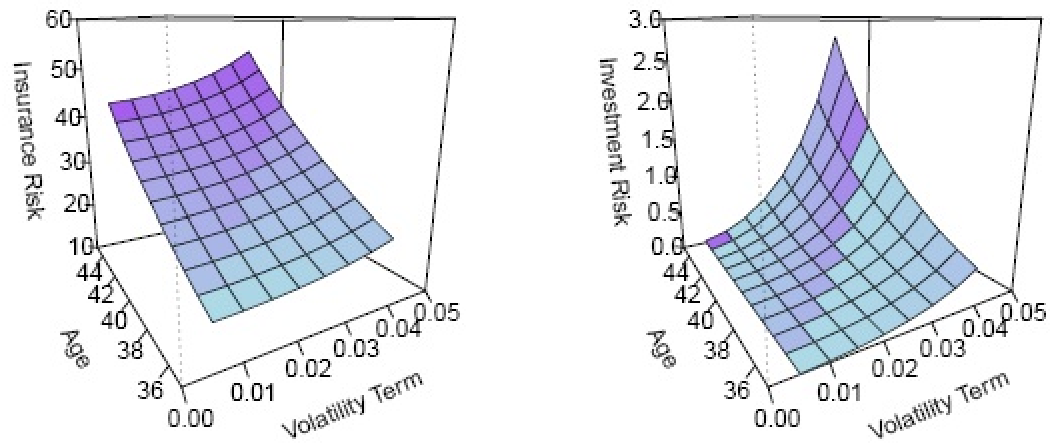

Figure 6.12: Insurance and Investment Risk vs σ_a and Age for a 15-year Disability Insurance Policy under the AR(1) Interest Rate Models ($\delta_0 = 0.06; \phi = 0.9$)

disability insurance portfolio with each of its subsidiary homogeneous portfolios shall be decomposed into the insurance risk and the investment risk.

Table 6.1 lists the information of the numerical example of the long-term disability insurance portfolio that we study. In total, this whole big portfolio consists of 14 homogeneous insurance portfolios with different entry ages, terms of policies, temporarily disability benefit amounts (TD Benefit), permanent disability benefit amounts (PD Benefit) and the lump sum death benefit amounts. The service table and the mortality table in Bowers et al.∼ (1997) were used to model the transition process between policyholder's statuses. The policyholders are assumed to be healthy at their entry ages.

Table 6.1: The Policy Information of the General Insurance Portfolio

Group No.	**No. of Policies**	**Age**	**Term of Policies**	**TD Benefit**	**PD Benefit**	**Death Benefit**
1	80	35	5	1,000	2,000	30,000
2	500	35	10	2,000	4,000	60,000
3	350	35	15	3,000	6,000	90,000
4	100	35	20	5,000	10,000	150,000
5	60	35	25	3,000	6,000	90,000
6	500	40	10	3,000	6,000	90,000
7	400	40	15	5,000	10,000	150,000
8	180	40	20	5,000	10,000	150,000
9	400	45	5	3,000	6,000	90,000
10	200	45	10	5,000	10,000	150,000
11	160	45	15	5,000	10,000	150,000
12	500	50	5	2,000	4,000	60,000
13	650	50	10	5,000	10,000	150,000
14	400	55	5	2,000	4,000	60,000

Now we are able to analyze the risks for each homogeneous insurance portfolio. Table 6.2 illustrates the numerical results of the two approaches of the insurance risk and the investment risk stated in Chapter 5. The top-left columns shown in the table are for $E\{Var[\frac{Z(c)}{c}|I]\}$, the insurance risk per policy calculated by conditioning on the interest rate (see (5.8)), and the bottom-left columns are for $Var\{E[\frac{Z(c)}{c}|M]\}$, the insurance risk per policy calculated by conditioning on the transition process (see (5.12)). The top-right columns in the table are for $Var\{E[\frac{Z(c)}{c}|I]\}$, the investment risk per policy calculated by conditioning on the interest rate (see (5.9)) and the bottom-right columns are for $E\{Var[\frac{Z(c)}{c}|M]\}$, the investment risk per policy calculated by conditioning on the transition process (see (5.11)).

Table 6.2: The Risk Decomposition for Each of Homogeneous Insurance Portfolios under Three Interest Rate Models

	$E\{Var[\frac{Z(c)}{c}\|I]\}$			$Var\{E[\frac{Z(c)}{c}\|I]\}$	
	Deterministic	**Binomial**	**AR(1)**	**Binomial**	**AR(1)**
Group 1	70,354	70,365	71,111	7	366
Group 2	94,109	94,241	96,341	1,579	17,696
Group 3	478,484	480,735	495,951	41,229	153,361
Group 4	6,402,452	6,471,241	6,724,840	680,755	1,082,959
Group 5	4,800,291	4,894,796	5,112,225	982,811	788,264
Group 6	326,467	326,933	334,348	8,818	98,341
Group 7	1,757,076	1,765,441	1,822,093	284,629	1,058,367
Group 8	5,224,336	5,280,491	5,488,758	1,640,440	2,634,408
Group 9	315,351	315,403	318,837	382	20,955
Group 10	3,549,058	3,554,128	3,635,256	62,963	706,025
Group 11	6,636,770	6,668,059	6,882,286	699,126	2,635,245
Group 12	179,497	179,526	181,471	443	24,413
Group 13	1,677,092	1,679,456	1,717,606	156,179	1,766,707
Group 14	344,719	344,776	348,528	1,103	60,827
Average	1,377,330	1,385,077	1,427,007	174,685	652,004
	$Var\{E[\frac{Z(c)}{c}\|M]\}$			$E\{Var[\frac{Z(c)}{c}\|M]\}$	
	Deterministic	**Binomial**	**AR(1)**	**Binomial**	**AR(1)**
Group 1	70,354	70,354	70,401	18	1,076
Group 2	94,109	94,108	94,543	1,712	19,494
Group 3	478,484	478,459	484,532	43,505	164,780
Group 4	6,402,452	6,401,668	6,555,695	750,328	1,252,105
Group 5	4,800,291	4,799,261	4,978,234	1,078,346	922,255
Group 6	326,467	326,465	327,986	9,287	104,703
Group 7	1,757,076	1,756,986	1,779,161	293,084	1,101,299
Group 8	5,224,336	5,223,713	5,346,714	1,697,218	2,776,452
Group 9	315,351	315,352	315,564	433	24,228
Group 10	3,549,058	3,549,034	3,565,365	68,057	775,916
Group 11	6,636,770	6,636,440	6,717,809	730,745	2,799,722
Group 12	179,497	179,497	179,616	472	26,268
Group 13	1,677,092	1,677,081	1,684,587	158,554	1,799,726
Group 14	344,719	344,720	344,947	1,159	64,408
Average	1,377,330	1,377,249	1,395,477	182,513	683,534

For most of the homogeneous insurance portfolio groups, the investment risk with the AR(1) model is much higher than the one with the binomial interest rate model. This is because the investment risk level is determined by the volatility terms in the two interest rate models that we select, i.e., σ_a and $\sigma_r(k)$. However, Group 5 including all the 25-year-term policies for 35-year-old policyholders is an exception case. This could be explained by the fact that the variations in the possible values of the annual interest rate under the binomial interest rate model are fairly high when the valuation term is long. Therefore, the binomial tree interest rate model is a good estimate for long-term valuations when the future market condition is unknown and the market return does not seem to have a clear trend of converging to a long-term mean.

Table 6.3: The Risk Decomposition for the Non-homogeneous Long-term Disability Insurance Portfolio under Three Interest Rate Models

	$E\left\{Var\left[\frac{Z(c)}{c}\|I\right]\right\}$			$Var\left\{E\left[\frac{Z(c)}{c}\|I\right]\right\}$	
No. of Policies	**Deterministic**	**Binomial**	**AR(1)**	**Binomial**	**AR(1)**
4,480	93,014	93,345	95,829	10,136.2	58,668
44,800	9,301.4	9,334.5	9,583	10,136.2	58,668
448,000	930.1	933.5	958.3	10,136.2	58,668
4,480,000	93.01	93.35	95.8	10,136.2	58,668
44,800,000	9.301	9.3	9.6	10,136.2	58,668
Infinity	0	0	0	10,136.2	58,668
	$Var\left\{E\left[\frac{Z(c)}{c}\|M\right]\right\}$			$E\left\{Var\left[\frac{Z(c)}{c}\|M\right]\right\}$	
No. of Policies	**Deterministic**	**Binomial**	**AR(1)**	**Binomial**	**AR(1)**
4,480	93,014	93,011	93,856	10,470.1	60,641
44,800	9,301.4	9,301.1	9,386	10,169.5	58,865
448,000	930.1	930.1	939	10,139.5	58,687
4,480,000	93.01	93.01	93.9	10,136.5	58,670
44,800,000	9.301	9.3	9.4	10,136.2	58,668
Infinity	0	0	0	10,136.2	58,668

Table 6.3 illustrates the numerical results for two approaches (see Chapter 5) of calculating the insurance risk and the investment risk under three interest rate models. Comparing with the numerical results calculated separately for each of small homogeneous portfolios, the insurance risk per policy for the big non-homogeneous portfolio is reduced to a large degree, from 1,377,330 in Table 6.2 to less than 100,000 in Table 6.3. This indicates that

the correlations among the policies of different homogeneous groups can never be neglected for valuation purposes. Thus, developing the methodology of evaluating the risk for the non-homogeneous long-term disability portfolio is crucial.

From Table 6.3, we observe that the insurance risk per policy is gradually fading out when the portfolio size is increasing. Therefore, the insurance risk per policy can be managed by pooling technique. However, the total riskiness per policy will not go to zero when there are infinite number of contracts in the portfolio. The limit of this risk is exactly the limit of the investment risk which can not be reduced by pooling.

Table 6.4: The Risk Decomposition Comparison for the Term Life Insurance Portfolio, Disability Only Portfolio and the Long-term Disability Insurance Portfolio under Three Interest Rate Models

			Total Insurance Risk		
	No. of Policies	**Benefits**	**Deterministic**	**Binomial**	**AR(1)**
Portfolio 1	4,480	Death only	335,825,056	335,815,917	338,391,200
Portfolio 2	4,480	Disabilities only	91,057,971	91,052,237	92,406,720
Sum of 1+2	8,960	N/A	426,883,027	426,868,154	430,797,920
Our Portfolio	4,480	Both	416,704,333	416,690,042	420,475,462
			Total Investment Risk		
	No. of Policies	**Benefits**	**Deterministic**	**Binomial**	**AR(1)**
Portfolio 1	4,480	Death only	0	21,922,374	142,059,187
Portfolio 2	4,480	Disabilities only	0	4,296,819	18,906,039
Sum of 1+2	8,960	N/A	0	26,219,193	160,965,226
Our Portfolio	4,480	Both	0	45,409,952	262,831,341

Recall that in Chapter 1, we have mentioned the advantage of designing this long-term disability insurance product is to better evaluate the insurance and investment risks for term life insurance portfolios and disability payment only portfolios under the circumstance that common policyholders exist in both of the portfolios. Table 6.4 compares the results of evaluating the risks of such portfolios separately and measuring the risks by regarding them as one whole portfolio. Extreme cases are considered: the policyholders involved in the term life insurance portfolio group and the disability payment only portfolio are exactly the same groups of people. Refer to (5.9) and (5.12) for the methodology of calculating the insurance risk and the investment risk. Instead of the average risk per policy, the total risk of the whole portfolio is applied.

From Table 6.4, we discover that the insurance risk is overestimated by about 2.44%-2.45%, and the investment risk is underestimated by approximately 40% once the valuation is done separately, neglecting the correlations of the two insurance groups. Therefore, taking into account of the correlation, we have the methodology to allow more accurate valuation results.

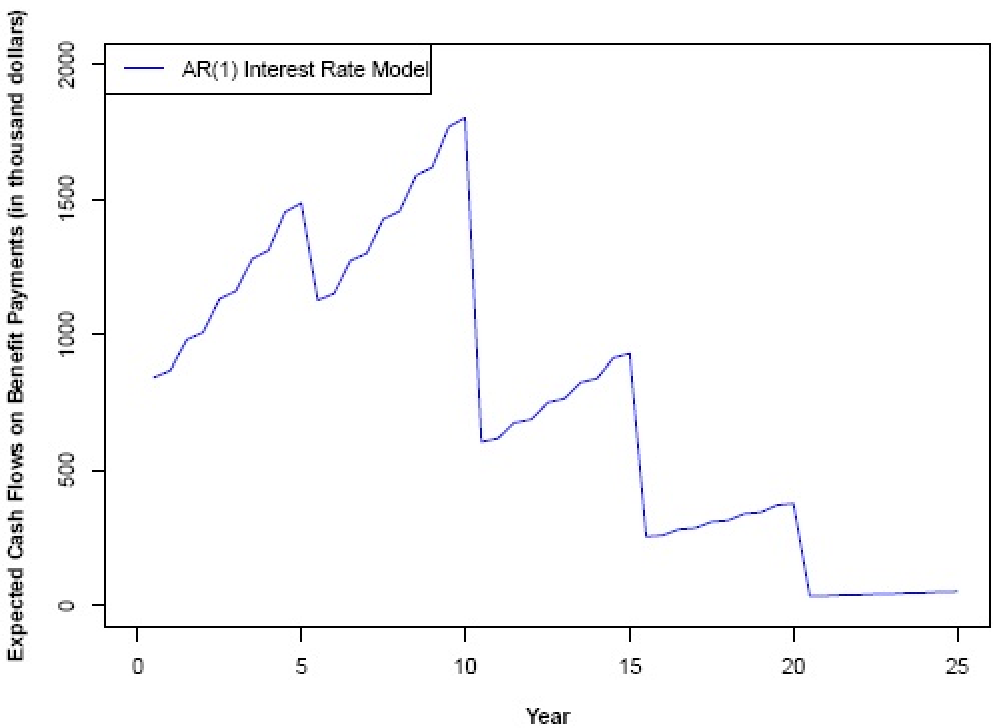

Figure 6.13: Expected Cash Flows of the General Long-term Disability Insurance Portfolio of Benefit Payments with AR(1) Interest Rate Models

Figure 6.13 shows the expected cash flow of benefit payments from the group effective date to the maturity date of the longest contracts in the insurance portfolio. We observe a 5-year cycle of the cash flow: the expected cash flow paid out is increasing within each cycle until it reaches the year that some of the policies terminate. The peak of the cash flow of benefit payments occurs at year 10 because the majority of the policies in the portfolio terminates at that time. Thus, appropriate hedging strategies considering duration matching possibly be made according to this cash flow pattern to avoid the ruin of the insurance company.

Chapter 7

Conclusion

In this project, we study an insurance product providing both a semiannual disability payment and a lump sum death benefit. By extending the methodology in Parker (1997) and Dickson et al.∼ (2009) to this new insurance product, we calculate the moments of future benefit streams and analyze the risks of such a non-homogeneous long-term disability insurance portfolio under two stochastic interest rate models, the AR(1) process and the binomial tree model. The deterministic interest rate model is also illustrated. The transition process between the four statuses of the policyholders is modeled by Markov chain techniques and this process is assumed to be independent among different policyholders. For the single policy case, we derive a recursive formula to calculate the first two moments of the PVFBP of the long-term disability insurance policy.

Two approaches by conditioning on the interest rate model and policyholder's transition process among four statuses, have been applied to evaluate the insurance risk and the investment risk of a long-term disability insurance portfolio, respectively. In practice, for an insurance portfolio with a large size, the insurance risk is relatively small compared with the investment risk by enlarging the portfolio size, since the former one fades out while the latter one reaches a limit. In addition, a good estimate of the investment risk is very significant for the insurers before bringing out any investment strategy. The accuracy in estimating the investment risk is thus the key issue to be considered in risk management.

According to our study, we observe that separate valuations for disability and life insurance products underestimate the investment risk if there are common active insureds who bought both products. The purpose of designing this new insurance product is to improve the accuracy in insurer's valuations of insurance portfolios including the life products and the disability products with insureds involved in both plans. Thus, the promotion of this product could help the insurers better manage the insurance risk and the investment risk, which are significant quantities of the risk management of insurance companies.

There are four quantities we mainly study in this project: the expectation and variance of the present value of future benefit streams, the insurance risk and the investment risk for the long-term insurance policies. Their sensitivities (single policy case) to the values of the long-term mean, the volatility and the convergence speed to the long-term mean parameter of the AR(1) model as well as the benefit amounts have been tested.

To sum up, all of the four quantities are increasing in the benefit payment amount, the volatility term σ_a, and the convergence speed ϕ in the AR(1) interest rate model. However, they are decreasing functions of the long-term mean parameter δ_0. The investment risk is very sensitive to the change in σ_a, especially for policyholders entering the plan at the senior age groups and at large values of σ_a. A similar interesting pattern also shows in the sensitivity test on the convergence speed ϕ: the investment risk suddenly increases when ϕ is close to one. For portfolio cases, we have discovered in an illustration a larger investment risk under the binomial tree interest rate model than the one under the AR(1) interest model when the policy term is 25 years. Furthermore, the design of this new insurance product gives rise to a much lower total insurance risk and higher investment risk to a large degree comparing with the one evaluated separately for the term life insurance portfolio and the disability insurance portfolio. Therefore, the advantage of this new product is to guarantee the insurer's valuation accuracy.

In our study, we only derive the formulas for the first two moments of the PVFBP of the long-term disability insurance policy under the Markov chain model without insurance data available. However, further research could be done based on the methodology and the primary result shown in this project. One could calculate the expectation of the difference between the PVFBP and the PVFPI (Present Value of Future Premium Income). Thus, cash flow projections of the whole insurance portfolio for 3-5 years could be done. Appropriate hedging strategies could be taken according to the cash flow pattern. As was indicated in Jones (1994), for some actuarial applications of the multi-state models, the Markov assumption is inappropriate. In the model that we study in this project, the probability of transition leaving from the disabled states may be affected not only by the age of the individual but also the time since becoming disabled, called duration. To incorporate this duration dependence, a semi-Markov chain process for the transitions can be used. See

Hoem (1972), Jones(1994) and Janssen and Manca (2007) for more details in using semi-Markov models in life insurance and disability insurance applications. Last but not least, the limitation we have in our research is that we have assumed all policyholders to be healthy at the moment of the effective date of the policy. However, this is not always true in practice. The methodology of calculating the transition probabilities illustrated in this thesis could be extended to allow different statues for the policyholders at the time of entry.

Bibliography

[1] J.A. Beekman. An alternative premium calculation method for certain long-term care coverages. *Actuarial Research Clearing House*, 2:179–190, 1990.

[2] F. Black, E. Derman, and W. Toy. A one-factor model of interest rates and its application to treasury bond options. *Financial Analysts Journal*, 46:33–39, 1990.

[3] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, and C.J. Nesbitt. *Actuarial Mathematics*. 2nd. ed. Schaumburg, Illinois: The Society of Actuaries, 1997.

[4] A.J.G. Cairns. *Interest Rate Models: An Introduction*. Princeton University Press, 2004.

[5] L. Chen. Analysis of joint life insurance with stochastic interest rates. Master's thesis, Simon Fraser University, 2010.

[6] J.W. Daniel. *Multi-state transition models with actuarial applications*. The Society of Actuaries, 2004. Study Manual.

[7] D. C. M. Dickson, M. Hardy, and H. R. Waters. *Actuarial Mathematics for Life Contingent Risks*. New York: Cambridge University Press, 2009.

[8] P. Gaillardetz. Valuation of life insurance products under stochastic interest rates. *Insurance: Mathematics and Economics*, 42:212–226, 2007.

[9] C. Giaccotto. Stochastic modelling ot interest rates: Actuarial vs. equilibrium approach. *The Journal of Risk and Insurance*, 53:435–453, 1986.

[10] S. Haberman, A. Olivieri, and E. Pitacco. Multiple state modelling and long term care insurance. Presented to the Staple Inn Actuarial Society, August 1997.

[11] S. Haberman and E. Pitacco. *Actuarial models for disability insurance*. London: Chapman and Hall, 1998.

[12] J.M. Hoem. Inhomogeneous Semi-Markov processes, select actuarial tables, and duration-dependence in demography. In *Population Dynamics*, pages 251–296. New York: Academic Press, 1972.

[13] J. Janssen and R. Manca. *Semi-Markov Risk Models for Finance, Insurance and Reliability*. New York: Springer Science+Business Media LLC, 2007.

[14] B.L. Jones. Actuarial calculations using a Markov model. *Transactions of the Society of Actuaries*, 10:395–404, 1994.

[15] B. Levikson and G. Mizrahi. Pricing long term care insurance contracts. *Insurance: Mathematics and Economics*, 14:1–18, 1994.

[16] E. Marceau and P. Gaillardetz. On life insurance reserves in a stochastic mortality and interest rates environment. *Insurance: Mathematics and Economics*, 25:261–280, 1999.

[17] J. Martin, H. Meltzer, and D. Elliot. *OPCS surveys of disability in Great Britain: Report 1 - The prevalence of disability among adults*. London: OPCS HMSO, 1988.

[18] S.M. Pandit and S. Wu. *Time Series and System Analysis with Applications*. New York: John Wiley & Sons, 1983.

[19] H.H. Panjer and D.R. Bellhouse. Stochastic modelling of interest rates with applications to life contigencies. *The Journal of Risk and Insurance*, 47:91–110, 1980.

[20] G. Parker. Stochastic analysis of the interaction between investment and insurance risks. *North American Actuarial Journal*, 1:55–84, 1997.

[21] D. Rajnes. Permanent disability social insurance programs in Japan. *Social Security Bulletin*, 70:61–84, 2010.

[22] C.M. Ramsay. The asymptotic ruin problem when the healthy and sick periods form an alternating renewal process. *Insurance: Mathematics and Economics*, 3:139–143, 1984.

[23] H.R. Waters. An approach to the study of multiple state models. *Journal of the Institute of Actuaries*, 111:363–374, 1984.

[24] H.R. Waters. The recursive calculations of the moments of the profit on a sickness insurance policy. *Insurance: Mathematics and Economics*, 9:101–113, 1990.